Vidya Jyothi Institute of Technology

(An Autonomous Institution)

(Accredited by NAAC, Approved by AICTE New Delhi & Permanently Affiliated to JNTUH)

Aziz Nagar Gate, C.B. Post, Hyderabad-500 075

Department of Humanities and Sciences

REGULATION : R18

BATCH : CSE D

ACADEMIC YEAR : 2019-20

PROGRAM : UG

YEAR/SEM : I/I

COURSE NAME : **MATRICES AND CALCULUS (MATHEMATICS I)**

COURSE CODE : A31002

COURSE COORDINATOR

H.O.D

NAME OF THE FACULTY: Dr. Kondala Rao Kanaparti

DESIGNATION: Associate Professor

Vidya Jyothi Institute of Technology

(An Autonomous Institution)

(Accredited by NAAC, Approved by AICTE New Delhi & Permanently Affiliated to JNTUH)

Aziz Nagar Gate, C.B. Post, Hyderabad-500 075

COURSE FILE INDEX

1. Vision and Mission of Institution
2. Vision and Mission of Department
3. POs, PSOs
4. Course Content
5. Course outcomes
6. Cos Mapping with Pos and PSOs
7. Student list
8. Time table
9. Lesson Plan
10. Academic Calendar
11. Course delivery Plan
12. Course Attainment
13. Mid I , MID-II Questions paper
14. Assignment I and II
15. Previous End semester Question papers
16. Course End Survey
17. Content Beyond Syllabus
18. Teaching Learning Methods
19. Course closure report
20. Course Material

1. VISION AND MISSION OF INSTITUTION

Vision of the Institution

- To develop into a reputed Institution at National and International level in Engineering, Technology and Management by generation and dissemination of knowledge through intellectual, cultural and ethical efforts with human values.
- To foster Scientific Temper in promoting the World class professional and technical expertise

Mission of the Institution

- To create state of art infrastructural facilities for optimization of knowledge acquisition.
- To nurture the students holistically and make them competent to excel in the global scenario.
- To promote R&D and Consultancy through strong Industry Institute Interaction to address the societal problems.

2. VISION AND MISSION OF DEPARTMENT

Vision of the Department

To produce the globally competent professionals in the field of Computer Science and Engineering

Mission of the Department

- To provide state-of-the-art facilities in Computer Science and Engineering, through innovative teaching learning practices
- To promote research and development in the frontier areas of Computer Science and Engineering and to work in interdisciplinary fields
- To enrich students with discipline and high integrity to serve the society and to inculcate the spirit of ethical values and leadership abilities
- To establish a collaborative environment between Industry and Academia

3. PROGRAMME OUTCOMES (PO'S) PROGRAMME SPECIFIC OUTCOMES (PSO'S)

&

PROGRAMME EDUCATIONAL OBJECTIVES (PEO'S)

Vidya Jyothi Institute of Technology

(An Autonomous Institution)

(Accredited by NAAC, Approved by AICTE New Delhi & Permanently Affiliated to JNTUH)

Aziz Nagar Gate, C.B. Post, Hyderabad-500 075

Programme Outcomes (PO's)

1. **Engineering knowledge:** Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization for the solution of complex engineering problems.
2. **Problem analysis:** Identify, formulate, research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design/development of solutions:** Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for public health and safety, and cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems:** Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.
5. **Modern tool usage:** Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools, including prediction and modeling to complex engineering activities, with an understanding of the limitations.
6. **The engineer and society:** Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal, and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability:** Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics:** Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.
9. **Individual and team work:** Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
10. **Communication:** Communicate effectively on complex engineering activities with the engineering community and with the society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance:** Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.
12. **Life-long learning:** Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

Program Specific Outcomes (PSO's)

PSO1: The ability to design and develop Algorithms to provide optimized solutions for societal needs

PSO2: Apply standard approaches and practices in Software Project Development through trending technologies

Program Educational Objectives (PEOs)

PEO1: Enhance the employablity of the graduates in Software industries/Public sector/Research organizations

PEO2: Acquire anlytical and computational abilities to pursue higher studies for professional growth

PEO3: Work in multidisciplinary project teams with effective communication skills and leadership qualities

PEO4: Develop professional ethics among the students and promote entrepreneurial abilities

4. SYLLABUS

Vidya Jyothi Institute of Technology

(An Autonomous Institution)

(Accredited by NAAC, Approved by AICTE New Delhi & Permanently Affiliated to JNTUH)

Aziz Nagar Gate, C.B. Post, Hyderabad-500 075

MATHEMATICS-I

I YEAR B.Tech, I SEMESTER	L	T/P/D	C
	3	1/-/-	4

(COMMON TO ALL BRANCHES)

Course Outcomes: After learning the contents of this paper the students must able to:

1. Write the matrix representation of system of linear equations and identify the consistency of the system of equations.
2. Find the Eigen values and Eigen vectors of the matrix and discuss the nature of the quadratic form.
3. Analyse the convergence of sequence and series.
4. Discuss the applications of mean value theorems to the mathematical problems, Evaluation of improper integrals using Beta and Gamma functions.
5. Examine the extrima of functions of two variables with/ without constraints.

UNIT-I: Matrices and Linear System of Equations

Matrices and Linear systems of equations: Real matrices – Symmetric, skew - symmetric, orthogonal, Linear Transformation – Orthogonal Transformation. Complex matrices: Hermitian, Skew – Hermitian and Unitary. Elementary row transformations-Rank-Echelon form, Normal form – Solution of Linear Systems – Direct Methods (Gauss Jordan).

UNIT-II: Eigen Values and Eigen Vectors

Eigen values, Eigen vectors – properties, Cayley-Hamilton Theorem (without Proof) - Inverse and powers of a matrix by Cayley-Hamilton theorem – Diagonolization of matrix- Quadratic forms: Nature, Index, Signature.

UNIT-III: Sequences & Series

Basic definitions of Sequences and series, Convergence and divergence, Ratio test, Comparison test, Integral test, Cauchy's root test, Raabe's test, Absolute and conditional convergence.

UNIT-IV: Beta & Gamma Functions and Mean Value Theorems

Gamma and Beta Functions-Relation between them, their properties – evaluation of improper integrals using Gamma / Beta functions.

Rolle's Theorem, Lagrange's mean value theorem, Cauchy's mean value theorem, Generalized Mean Value theorem (all theorems without proof) – Geometrical interpretation of Mean value theorems.

UNIT-V: Functions of several variables

Partial Differentiation and total differentiation, Functional dependence, Jacobian Determinant-Maxima and Minima of functions of two variables with constraints and without constraints, Method of Lagrange Multipliers.

TEXT BOOKS
&
REFERENCE BOOKS

Text Books & Reference Books

Text Books	
1	B.S. Grewal, Higher Engineering Mathematics, Khanna Publishers, 36th Edition, 2010
2	Advanced Engineering Mathematics by Jain &Iyengar Narosa Pulications
3	Ramana B.V., Higher Engineering Mathematics, Tata McGraw Hill New Delhi, 11th Reprint, 2010.
Reference Books	
1	G.B. Thomas and R.L. Finney, Calculus and Analytic geometry, 9th Edition, Pearson, Reprint, 2002
2	Erwin Kreyszig, Advanced Engineering Mathematics, 9th Edition, John Wiley & Sons, 2006.
3	Srimanta Pal and Subodh C. Bhunia, Engineering Mathematics, Oxford University Press, 2015.
Other Resources	
1	www.geocites.com
2	www.mathforum.org
3	www.mathworld.wolfram.com
4	www.iitb.ac.in
5	www.iitk.ac.in
6	www.iitm.ac.in
7	www.eduinstitutions.com
8	www.isical.ac.in

5. COURSE OUTCOMES

Vidya Jyothi Institute of Technology

(An Autonomous Institution)

(Accredited by NAAC, Approved by AICTE New Delhi & Permanently Affiliated to JNTUH)

Aziz Nagar Gate, C.B. Post, Hyderabad-500 075

Course Outcomes:

At the end of the course, the students should be able to:

CO1 Unit1	Write the matrix representation of system of linear equations and identify the consistency of the system of equations.
CO2 Unit2	Find the Eigen values and Eigen vectors of the matrix and discuss the nature of the quadratic form.
CO3 Unit3	Analyse the convergence of sequence and series.
CO4 Unit4	Discuss the applications of mean value theorems to the mathematical problems, Evaluation of improper integrals using Beta and Gamma functions.
CO5 Unit5	Examine the extrima of functions of two variables with/ without constraints.

6. MAPPING OF COURSE OUTCOMES WITH PO'S & PSO'S

Vidya Jyothi Institute of Technology

(An Autonomous Institution)

(Accredited by NAAC, Approved by AICTE New Delhi & Permanently Affiliated to JNTUH)

Aziz Nagar Gate, C.B. Post, Hyderabad-500 075

Mapping of Course outcomes with PO's & PSO's

Course Outcomes	PO's	PSOs
CO1	PO1, PO2, PO3, PO4, PO5, PO6, PO9, P10, PO11 & PO12	PSO1 & PSO2
CO2	PO1, PO2, PO3, PO4, PO5, PO6, PO9, P10, PO11 & PO12	PSO1 & PSO2
CO3	PO1, PO2, PO3, PO4, PO5, PO6, PO9, P10, PO11 & PO12	PSO1 & PSO2
CO4	PO1, PO2, PO3, PO4, PO5, PO6, PO9, P10, PO11 & PO12	PSO1 & PSO2
CO5	PO1, PO2, PO3, PO4, PO5, PO6, PO9, P10, PO11 & PO12	PSO1 & PSO2

Articulation matrix of Course outcomes with PO's

Course Name: Mathematics-I														
	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12	PSO1	PSO2
CO1	3	3	3	3	3	2	-	-	1	1	2	2	3	2
CO2	3	3	3	3	3	2	-	-	1	1	2	2	3	2
CO3	3	3	3	3	3	2	-	-	1	1	2	2	3	2
CO4	3	3	3	3	3	2	-	-	1	1	2	2	3	2
CO5	3	3	3	3	3	2	-	-	1	1	2	2	3	2
AVERAGE	3	3	3	3	3	2	-	-	1	1	2	2	3	2

7. STUDENT LIST 2019-20

Vidya Jyothi Institute of Technology

(An Autonomous Institution)

(Accredited by NAAC, Approved by AICTE New Delhi & Permanently Affiliated to JNTUH)

Aziz Nagar Gate, C.B. Post, Hyderabad-500 075

Branch and Section: CSE D

S.No	H.T.NO	NAME	S.No	H.T.NO	NAME
1	19911A05J1	Aakash Kumar	31	19911A05M1	Kurva Siva Kumar
2	19911A05J2	Agiru Nikhil	32	19911A05M2	M Naveen
3	19911A05J3	Anugu Sridevi	33	19911A05M3	Malkapuram Pooja
4	19911A05J4	Arcot Ravi Teja	34	19911A05M4	Mallela Sushanth
5	19911A05J5	Avusula Pujitha	35	19911A05M5	M Ajay Kumar
6	19911A05J6	B Lakshmi Sirisha	36	19911A05M6	M Ganesh S S S
7	19911A05J7	B Sudheer Kumar	37	19911A05M7	Nalam Sangamithra
8	19911A05J8	Bandi Vyshnavi	38	19911A05M8	Nemmani Sathvik Rao
9	19911A05J9	Boddu Shiva Kumar	39	19911A05M9	P Shashi Rekha
10	19911A05K0	Buchupalli Mohitha	40	19911A05N0	Peddinti Arunjyothi
11	19911A05K1	Ch Paramesh	41	19911A05N1	P G Venkata Ashok
12	19911A05K2	Challa Sai Stish	42	19911A05N2	P Yashwanth Goud
13	19911A05K3	Chettukindi Sathwik	43	19911A05N3	Pusala Vijaykumar
14	19911A05K4	Darmoju Deekshitha	44	19911A05N4	Rajvardhan Chirag
15	19911A05K5	Dharavath Bhoomika	45	19911A05N5	Ravula Sai Laxmi
16	19911A05K6	D Uma Maheshwar Rao	46	19911A05N6	S V Dinesh Reddy
17	19911A05K7	Gadarla Praveen	47	19911A05N7	Sathe Ruchitha
18	19911A05K8	G Bhavya Sri	48	19911A05N8	Seepathi Chakrapani
19	19911A05K9	Golla Anushya Kirthy	49	19911A05N9	S Advaith Sai
20	19911A05L0	I N Shivashankar	50	19911A05P0	Tadepalli Saketh
21	19911A05L1	Jagannathula Vandana	51	19911A05P1	Thasleem
22	19911A05L2	Jala Mallesh Yadav	52	19911A05P2	T Gowtham Reddy
23	19911A05L3	Jammugani Thanish	53	19911A05P3	Vaghmare Vishal
24	19911A05L4	K P Sai Krishna Reddy	54	19911A05P4	Vasam Vishal
25	19911A05L5	K Vaishnavi	55	19911A05P5	Vislavath Shivakumar
26	19911A05L6	Kadali Hemanjalai	56	19911A05P6	Y Keerthi
27	19911A05L7	Katakam Varsha	57	19911A05P7	Y Sanjana Reddy
28	19911A05L8	Konkimalla Tarun	58	19911A05P8	Yarram Setty Adarsh
29	19911A05L9	Kothamasi Swetha	59	19911A05P9	Y Siva Satwik Reddy
30	19911A05M0	K Vamshinath Reddy	60	19911A05Q0	N Yaswanth Varma

HOD

8. TIME TABLE

Vidya Jyothi Institute of Technology

(An Autonomous Institution)

(Accredited by NAAC, Approved by AICTE New Delhi & Permanently Affiliated to JNTUH)

Aziz Nagar Gate, C.B. Post, Hyderabad-500 075

Time Table

Class Hour	1	2	3	4		5	6	7
Time	9:00 – 9:55	9.55 – 10:50	10:50– 11:45	11.45 – 12.40		1:25 – 2:20	2:20 – 3:15	3:15 – 4:05
MON				CSE D	12:40 – 1:25 LUNCH BREAK			
TUE				CSE D				
WED	CSE D							
THU							CSE D	
FRI		CSE D	CSE D					
SAT		CSE D						

HOD

9. LESSON PLAN

Vidya Jyothi Institute of Technology

(An Autonomous Institution)

(Accredited by NAAC, Approved by AICTE New Delhi & Permanently Affiliated to JNTUH)

Aziz Nagar Gate, C.B. Post, Hyderabad-500 075

Lecture Plan

S. No.	Topic	No of Class required	Total no of Classes	Teaching Learning Process	Text Book 1	Text Book II
UNIT-I Matrices and Linear System of Equations						
1	Matrices-Introduction	2	14	Blackboard and chalk and PPT	26,27,28	3.1 to 3.5
2	Real Matrices-Symmetric, Skew-Symmetric & Orthogonal Matrices	1			30 to 34	3.6, 3.7 & 3.78 to 3.80
3	Complex Matrices-Hermitian, Skew-hermitian & Unitary Matrices	1			68 to 71	3.6, 3.7 & 3.78 to 3.80
4	Elementary row transformations-Rank-Echelon form	1			35 to 40	3.50 to 3.51
5	Rank-Normal form	1			35 to40	
6	Solutions of system of linear equations	1			43 to 45	
7	Consistency of system of non- homogeneous linear equations	2			46 to 48	
8	system of homogeneous linear equations	2			48 to 51	
9	Solutions of system of equations by Gauss Jordan method	1				18.28
10	Solutions of system of equations by LU Decomposition method	1				18.29 to 18.32
11	Revision	1				
Unit-II: Eigen Values and Eigen Vectors						
1	Eigen values and Eigen vectors Introduction	2			54	3.62

2	Properties	2	14	Blackboard and chalk PPT	57 to 58	3.63 to 3.64
3	Eigen values and Eigen vectors Problems	1			58	3.67 to 3.70
4	Eigen values and Eigen vectors Problems	1			60	3.67 to 3.70
5	Cayley-Hamilton theorem(with out proof)	1			58	3.65
6	Inverse and powers of a matrix by C-H theorem	1			59 to61	3.66
7	Digitalization of Matrix	2			61 to 64	3.71 to 3.76
8	Quadratic forms	1			64 to 67	3.81
9	Reduction of Quadratic form to Canonical form by Orthogonal Transformation	1			65	3.82
10	Nature, Index and Signature of Quadratic Form.	1			66 & 67	3.82
11	Revision	1				
Unit-III: Sequences & Series						
1	**Sequences & Series Introduction**	2	10	Blackboard and chalk	**365**	12.1
2	Basic definitions of Sequences and series	1			365&366	12.2 to 12.3, 12.13
3	Convergence and divergence	1			365&366	12.14, 12.15
4	Ratio test	1			373 to377	12.17 to 12.18
5	Comparison test	1			368 &371	12.16
6	Cauchy's root test	1			380 & 381	12.19, 12.20
7	Raabe's test	1			377 to 379	12.17 to 12.18
8	Integral test	1			369	12.19, 12.20
9	Absolute and conditional convergence	1			382 to 385	12.20 to

						12.23
10	Revision	1				
Unit-IV: Beta & Gamma Functions and Mean Value Theorems						
1	**Beta & Gamma Functions introduction**	1	12	Blackboard and chalk	302	1.49 to 1.50
2	Gamma and Beta Functions	2			302	1.60 to 1.62
3	Relation between them, their properties	2			303	1.63 & 1.64
4	evaluation of improper integrals using Gamma / Beta functions	2			305 to310	1.65 to 1.68
5	**Mean Value Theorems**	1			142	1.10
6	Rolle's Theorem	1			142 to 144	1.11
7	Lagrange's mean value theorem	1			142 to 144	1.12 & 1.13
8	Cauchy's mean value theorem	1			144 to 144	1.12 & 1.14
9	Geometrical interpretation of Mean value theorems	1			142 to 144	1.15
10	Generalized Mean Value theorem (all theorems without proof)	1			145 to151	1.19 to 1.22
11	Revision	1				
Unit-V: Functions of several variables						
1	**Functions of several variables**	**1**	**10**	Blackboard and chalk	197	2.1 to 2.7
2	Partial Differentiation and total differentiation	**2**			198 to 213	2.12 to 3.31
3	Functional dependence	**1**			214	2.26 to 2.28
4	Jacobian Determinant	**1**			215 to 218	2.26 to 2.28
5	Maxima and Minima of functions of two variables	**2**			226	2.46 to

	with constraints and without constraints					2.50
6	Method of Lagrange Multipliers	**2**		Blackboard and chalk	229 to 233	2.51 to 2.54
12	Revision	**2**		Blackboard and chalk		
13	Revision for all	**2**		Blackboard and chalk		
Total No of classes: 60						

HOD

10. ACADEMIC CALENDAR

VIDYA JYOTHI INSTITUTE OF TECHNOLOGY (AUTONOMOUS)

(Accredited by NAAC & NBA, Approved By A.I.C.T.E., New Delhi, Permanently Affiliated to JNTU, Hyderabad)
(Aziz Nagar, C.B.Post, Hyderabad -500075)

ACADEMIC CALENDAR FOR I B.Tech I & II SEMESTER FOR THE YEAR 2019-20

FIRST SEMESTER		Commencement of Class work : **02.08.2019**	
	FROM	**TO**	**DURATION**
Orientation Programme	02.08.2019	17.08.2019	2 WEEKS
I Spell of Instruction	19.08.2019	28.09.2019	6 WEEKS
I Mid Examinations	30.09.2019	04.10.2019	4 DAYS
Dussehra Holidays	05.10.2019	13.10.2019	9 DAYS
II Spell of Instructions	14.10.2019	07.12.2019	8 WEEKS
II Mid Examinations	09.12.2019	12.12.2019	4 DAYS
Practical Examinations	13.12.2019	19.12.2019	6 DAYS
III Mid Examinations	20.12.2019	21.12.2019	2 DAYS
End Semester Examinations	23.12.2019	04.01.2020	2 WEEKS
SECOND SEMESTER		Commencement of Class work : **06.01.2020**	
I Spell of Instruction	06.01.2020	10.01.2020	5 DAYS
Sankranthi Holidays	11.01.2020	15.01.2020	5 DAYS
Technical/Sports Fest	16.01.2020	18.01.2020	3 DAYS
I Spell of Instruction Continuation	20.01.2020	09.03.2020	7 WEEKS 1 DAY
I Mid Examinations	11.03.2020	14.03.2020	4 DAYS
II Spell of Instruction	16.03.2020	09.05.2020	8 WEEKS
II Mid Examinations	11.05.2020	14.05.2020	4 DAYS
Practical Examinations	15.05.2020	21.05.2020	6 DAYS
III Mid Examinations	22.05.2020	23.05.2020	2 DAYS
End Semester/Supplementary Examinations	27.05.2020	06.06.2020	10 DAYS
Summer Vacation	08.06.2020	13.06.2020	1 WEEK

Commencement of class work for II Year I Sem will be from 15-06-2020

Vidya Jyothi Institute of Technology, Himayatnagar Vill., C.B.Post, Hyderabad-75

DIRECTOR

11. COURSE DELIVERY PLAN

Vidya Jyothi Institute of Technology

(An Autonomous Institution)

(Accredited by NAAC, Approved by AICTE New Delhi & Permanently Affiliated to JNTUH)

Aziz Nagar Gate, C.B. Post, Hyderabad-500 075

S. No.	Name of the Topic	Expected date of Completion	Date of Completion	Teaching Aid/ Teaching Methodology
UNIT – 1: Matrices and Linear System of Equations				
1	Matrices-Introduction	09-08-2019, 13-08-2019	09-08-2019, 13-08-2019	Chalk and Talk
2	Real Matrices-Symmetric, Skew-Symmetric & Orthogonal Matrices	14-08-2019, 16-08-2019, 17-08-2019	14-08-2019, 16-08-2019, 17-08-2019	Chalk and Talk
3	Complex Matrices- Hermitian, Skew-hermitian & Unitary Matrices	19-08-2019	19-08-2019	Chalk and Talk
4	Elementary row transformations-Rank-Echelon form	20-08-2019, 21-08-2019	20-08-2019, 21-08-2019	Chalk and Talk
5	Rank-Normal form	22-08-2019	22-08-2019	Chalk and Talk
6	Solutions of system of linear equations	23-08-2019	23-08-2019	Chalk and Talk
7	Consistency of system of non-homogeneous linear equations	27-08-2019	27-08-2019	Chalk and Talk
8	system of homogeneous linear equations	28-08-2019	28-08-2019	Chalk and Talk
9	Solutions of system of equations by Gauss Jordan method	29-08-2019, 30-08-2019	29-08-2019, 30-08-2019	Chalk and Talk
10	Solutions of system of equations by LU Decomposition method	03-09-2019	03-09-2019	Chalk and Talk
11	Revision			Chalk and Talk
UNIT – 2: Eigen Values and Eigen Vectors				
12	Eigen values and Eigen vectors Introduction	04-09-2019	11-09-2019,	Chalk and Talk
13	Properties	06-09-2019	13-09-2019,	Chalk and Talk Video Presentation
14	Eigen values and Eigen vectors Problems	07-09-2019	16-09-2019,	Chalk and Talk
15	Eigen values and Eigen vectors Problems	11-09-2019,	17-09-2019,	Chalk and Talk Video

				Presentation
16	Cayley-Hamilton theorem(without proof)	13-09-2019	18-09-2019	Chalk and Talk Video Presentation
17	Inverse and powers of a matrix by C-H theorem	16-09-2019	20-09-2019 23-09-2019	Chalk and Talk Video Presentation
18	Digitalization of Matrix	17-09-2019 18-09-2019,	24-09-2019 26-09-2019	Chalk and Talk
19	Quadratic forms	20-09-2019	30-09-2019	Book I and Book II
20	Reduction of Quadratic form to Canonical form by Orthogonal Transformation	23-09-2019 24-09-2019	23-10-2019	Book I and Book II
21	Nature, Index and Signature of Quadratic Form	26-09-2019	25-10-2019	Book I and Book II
22	Revision	30-09-2019	25-10-2019	Book I and Book II
UNIT – 3: Sequences & Series				
23	**Sequences & Series Introduction**	23-10-2019	26-10-2019	Chalk and Talk
24	Basic definitions of Sequences and series	23-10-2019	28-10-2019	Chalk and Talk
25	Convergence and divergence	25-10-2019	29-10-2019	Chalk and Talk
26	Ratio test	26-10-2019 28-10-2019	31-10-2019 01-11-2019	Chalk and Talk
27	Comparison test	29-10-2019	02-11-2019	Chalk and Talk
28	Raabe's test	31-10-2019	04-11-2019	Chalk and Talk
29	Cauchy's root test	01-11-2019	05-11-2019	Chalk and Talk
30	Integral test	02-11-2019	06-11-2019	Chalk and Talk
31	Absolute and conditional convergence	04-11-2019	07-11-2019	Chalk and Talk
32	Revision	05-11-2019	08-11-2019	Chalk and Talk
UNIT – 4: Beta & Gamma Functions and Mean Value Theorems				
33	**Beta & Gamma Functions**	06-11-2019	11-11-2019	Chalk and Talk
34	Gamma and Beta Functions	07-11-2019 08-11-2019	13-11-2019, 14-11-2019	Chalk and Talk
35	Relation between them, their properties	11-11-2019,	15-11-2019,	Chalk and

		13-11-2019	16-11-2019	Talk
36	Evaluation of improper integrals using Gamma / Beta functions	14-11-2019	18-11-2019	Chalk and Talk
37	**Mean Value Theorems**	15-11-2019	18-11-2019	Chalk and Talk
38	Rolle's Theorem	16-11-2019	19-11-2019	Chalk and Talk
39	Lagrange's mean value theorem	18-11-2019	20-11-2019	Chalk and Talk
40	Cauchy's mean value theorem	18-11-2019	20-11-2019	Chalk and Talk
41	Geometrical interpretation of Mean value theorems	15-11-2019 16-11-2019 18-11-2019	16-11-2019 18-11-2019	Chalk and Talk
42	Generalized Mean Value theorem (all theorems without proof)	19-11-2019	21-11-2019	Chalk and Talk
43	Revision	19-11-2019	22-11-2019	Chalk and Talk
UNIT – 5: Functions of several variables				
44	**Functions of several variables**	20-11-2019	23-11-2019	Chalk and Talk
45	Partial Differentiation and total differentiation	20-11-2019	25-11-2019	Chalk and Talk
46	Functional dependence	21-11-2019	25-11-2019	Chalk and Talk
47	Jacobian Determinant	22-11-2019	26-11-2019	Chalk and Talk
48	Maxima and Minima of functions of two variables with constraints and without constraints	23-11-2019, 25-11-2019	27-11-2019	Chalk and Talk
49	Method of Lagrange Multipliers	26-11-2019, 27-11-2019	28-11-2019, 29-11-2019	Chalk and Talk
50	Revision	28-11-2019,	30-11-2019	Chalk and Talk
51	Revision of all units	29-11-2019, 30-11-2019, 02-12-2019, 03-12-2019, 04-12-2019, 05-12-2019	02-12-2019, 03-12-2019, 04-12-2019, 05-12-2019	Chalk and Talk

12. COURSE ATTAINMENT

Vidya Jyothi Institute of Technology

(An Autonomous Institution)

(Accredited by NAAC, Approved by AICTE New Delhi & Permanently Affiliated to JNTUH)

Aziz Nagar Gate, C.B. Post, Hyderabad-500 075

Course Assessment Sheet

Batch : **I B. Tech**

Academic Year/Sem : 2019-2020

Course Name : **Mathematics I**

Course Number : **A31002**

Course Attainment = 80% of Direct + 20% of Indirect

Remarks and suggestions:

1 - Slight 2 - Moderate 3 – Substantial

Vidya Jyothi Institute of Technology(Autonomous)

(Accredited by NAAC, Approved By A.I.C.T.E., New Delhi, Permanently Affiliated to JNTU, Hyd)

(Aziz Nagar, C.B.Post, Hyderabad -500075)

Department of Humanities and Sciences

ATTAINMENT SHEET

Academic Year: 2019-2020 **Semester-I** **Regulation: R-18**

Course Name: Mathematics-I **Course Code:A21002**

Name of the Faculty: Dr. Kondala Rao Kanaparti **Branch: CSE D**

S.No	Reg.No	MID I Threshold 60%									MID II Threshold 60%									Threshold
				PART-A						PART-B			PART-A						PART-B	
		ASM - I (5M)	DESCRIPTIVE	Q1 (2M)	Q2 (2M)	Q3 A (1M)	Q3 B (1M)	Q4 (5M)	Q5 (5M)	Q6 (4M)	ASM - II (5M)	DESCRIPTIVE	Q1 (2M)	Q2 (2M)	Q3 A (1M)	Q3 B (1M)	Q4 (4M)	Q5 (5M)	Q6 (5M)	End Exam (75M)
1	19911A05J1	5	20	2	2	1	1	5	5	4	5	18	2	2	1	1	4	4	4	71
2	19911A05J2	5	19	2	2	1	1	4	5	4	5	15	1	2	1	1	4	3	3	35
3	19911A05J3	5	20	2	2	1	1	5	5	4	5	19	2	1	1	1	4	5	5	70
4	19911A05J4	4	12	1	1	1	1	3	3	2	4	7	2	2	1		2			18
5	19911A05J5	5	19	2	1	1	1	5	5	4	5	20	2	2	1	1	4	5	5	71
6	19911A05J6	5	20	2	2	1	1	5	5	4	5	20	2	2	1	1	4	5	5	72
7	19911A05J7	5	20	2	2	1	1	5	5	4	5	18	2	2		1	1	5	5	64
8	19911A05J8	5	16	2		1	1	4	4	2	5	12	2		1		3	4	2	47
9	19911A05J9	3	17	1	2	1	1	4	5	3	3	18	2	2	1	1	4	4	4	54
10	19911A05K0	5	19	2	2	1	1	5	4	4	5	19	2	2		1	4	5	5	55
11	19911A05K1	5	20	2	2	1	1	5	5	4	5	18	1	1	1	1	4	5	5	50
12	19911A05K2	5	13		1	1	1	5	5		5	10						5	5	17
13	19911A05K3	5	20	2	2	1	1	5	5	4	5	20	2	2	1	1	4	5	5	66
14	19911A05K4	5	20	2	2	1	1	5	5	4	5	20	2	2	1	1	4	5	5	69
15	19911A05K5	5	20	2	2	1	1	5	5	4	5	20	2	2	1	1	4	5	5	71
16	19911A05K6	0	19	2	2	1	1	5	5	3	0	12				1		3	4	53
17	19911A05K7	5	20	2	2	1	1	5	5	4	5	20	2	2	1	1	4	5	5	72
18	19911A05K8	5	19	2	1	1	1	5	5	4	5	15	2		1		4	4	4	54

19	19911A05K9	5	20	2	2	1	1	5	5	4	5	17		2	1	1	3	5	5	60
20	19911A05L0	5	20	2	2	1	1	5	5	4	5	19	2	2		1	4	5	5	58
21	19911A05L1	5	18	2	2	1		5	5	3	5	14	2		1		3	3	3	49
22	19911A05L2	5	15	2	2	1	1	5		4	5	16	2	2	1	1		5	5	45
23	19911A05L3	5	20	2	2	1	1	5	5	4	5	20	2	2	1	1	4	5	5	69
24	19911A05L4	5	18	2	2	1	1	4	4	4	5	16		1	1	1	4	5	4	38
25	19911A05L5	5	13	2		1	1		5	4	5	10	2		1	1	3		3	28
26	19911A05L6	5	18	2	2			5	5	4	5	14	2	2	1		4	5		39
27	19911A05L7	5	16	1	1	1	1	4	4	4	5	16	2	2	1	1		5	5	41
28	19911A05L8	5	19	1	2	1	1	5	5	4	5	17	1	1		1	4	5	5	62
29	19911A05L9	5	17		2	1		5	5	4	5	17	2		1	1	4	4	5	48
30	19911A05M0	5	10	2	2	1	1			4	5	12		2				5	5	4
31	19911A05M1	5	15	2	2	1	1		5	4	5	16	2	2			4	4	4	28
32	19911A05M2	5	5					5	5		5	9			1		3		5	15
33	19911A05M3	5	19	2	2	1	1	5	4	4	5	19	1	2	1	1	4	5	5	62
34	19911A05M4	4	20	2	2	1	1	5	5	4	4	19	2	1	1	1	4	5	5	63
35	19911A05M5	5	20	2	2	1	1	5	5	4	5	20	2	2	1	1	4	5	5	62
36	19911A05M6	4	16	1	1	1		5	5	3	4	14					4	5	5	46
37	19911A05M7	5	18	2	2	1	1	4	4	4	5	20	2	2	1	1	4	5	5	61
38	19911A05M8	5	20	2	2	1	1	5	5	4	5	20	2	2	1	1	4	5	5	62
39	19911A05M9	5	17		2	1	1	5	5	3	5	17	2		1	1	4	5	4	55
40	19911A05N0	5	18	2		1	1	5	5	4	5	18	2	2	1	1	4	4	4	54
41	19911A05N1	5	20	2	2	1	1	5	5	4	5	20	2	2	1	1	4	5	5	68
42	19911A05N2	5	8	1	2	1				4	5	8	2			1			5	--
43	19911A05N3	5	12	2		1	1	4		4	5	15	2	1	1	1		5	5	30
44	19911A05N4	3	16				1	5	5	4	3	10					4	3	3	28
45	19911A05N5	5	14	2	1		1	5	5		5	17	1	2		1	4	5	4	38
46	19911A05N6	5	20	2	2	1	1	5	5	4	5	20	2	2	1	1	4	5	5	55
47	19911A05N7	5	18	2	2	1	1	4	4	4	5	17	1	1	1	1	4	4	5	59
48	19911A05N8	5	18	1	2	1	1	5	5	3	5	20	2	2	1	1	4	5	5	63
49	19911A05N9	4	19	2	1	1	1	5	5	4	4	15	2		1		4	4	4	63
50	19911A05P0	5	18	2	2	1	1	4	4	4	5	18		2	1	1	4	5	5	56
51	19911A05P1	5	20	2	2	1	1	5	5	4	5	20	2	2	1	1	4	5	5	70

52	19911A05P2	5	5				1			4	5	4	2	2						11
53	19911A05P3	5	19	2	2	1	1	4	5	4	5	17	1	1	1	1	3	5	5	48
54	19911A05P4	5	20	2	2	1	1	5	5	4	5	18	2	2	1	1	4	4	4	54
55	19911A05P5	5	17	1	2	1	1	3	5	4	5	17	2	1			4	5	5	55
56	19911A05P6	5	20	2	2	1	1	5	5	4	5	20	2	2	1	1	5	5	4	66
57	19911A05P7	5	20	2	2	1	1	5	5	4	5	18	2	2	1	1	4	4	4	58
58	19911A05P8	5	12	2			1		5	4	5	16	2	2	1	1		5	5	26
59	19911A05P9	5	12		2	1	1		5	3	5	5	2	2	1					26
60	19911A05Q0	3	16	2	2	1	1	4	5	3	3	14	2	2			4	4	4	26
Average marks		4.8	17	1.8	1.8	1.0	1.0	4.7	4.8	3.8	4.75	16.17	1.9	1.8	1.0	1.0	3.8	4.6	4.6	50.1
No of students		60	60	53	52	54	54	53	55	57	60	60	51	47	45	44	50	54	56	59
%of students		98.3	100	84.91	84.62	100.00	100.00	#####	#####	96.49	98.33	100	86.27	80.85	100.00	100.00	96.00	100.00	98.21	71.19
CO ATTAINMENT		3	3	3.0	3.0	3.0	3.0	3.0	3.0	3.0	3	3	3.0	3.0	3.0	3.0	3.0	3.0	3.0	3.0

ASSESSMENT OF COs FOR THE COURSE

CO	Method	value	Avg	CO Attainment (Internal)	CO Attainment (End Exam)	Overall CO Attainment
CO 1	ASM I	3	3.0			
	MID I - PART A - Q1	3.0				
	MID I - PART A - Q3 A	3.0				
	MID I - PART B - Q4	3.0				
	ASM I	3				
	MID I - PART A - Q2	3.0				

CO	MID I - PART A - Q3 B	3.0	3.0			
	MID I - PART B - Q5	3.0				
CO	ASM I	3	3.0	3.00	3.00	3.00
	ASM II	3.0				
	MID I - PART B - Q6	3.0				
	MID II - PART B - Q4	3.0				
CO 4	ASM II	3	3.0			
	MID II - PART A - Q1	3.0				
	MID II - PART A - Q3 A	3.0				
	MID II - PART B - Q5	3.0				
CO 5	ASM II	3	3.0			
	MID II- PART A - Q2	3.0				
	MID II - PART A - Q3 B	3.0				
	MID II - PART B - Q6	3.0				

13. MID I, MID II QUESTION PAPERS (CO, PO, BL)

VidyaJyothi Institute of Technology(Autonomous)

(Accredited by NBA & NAAC Approved By A.I.C.T.E., New Delhi, Permanently Affiliated to JNTU, Hyd)

(Aziz Nagar, C.B.Post, Hyderabad -500075)

Question Paper Title: I B.Tech I Semester Question Paper for Mid Exam I		
Total Duration (H:M):1:30	Course: Matrices and Calculus (Mathematics-I)	Maximum Marks:20
Branch : All Branches	Session : FN	Date:01-10-2019

S.No	Course Outcomes
1	Write the matrix representation of system of linear equations and identify the consistency of the system of equations.
2	Find the Eigen values and Eigen vectors of the matrix and discuss the nature of the Quadratic form
3	Analyze the convergence of sequence and series
4	Discuss the applications of mean value theorems to the mathematical problems, Evaluation of improper integrals using Beta and Gamma functions
5	Examine the extrema of functions of two variables with or without constraints.

Blooms level	
Remember	I
Understand	II
Apply	III
Analyze	IV
Evaluate	V
Create	VI

Q.No	Questions	Marks	CO	PO	II
	PART-A $3 \times 2 = 6M$				
	ANSWER ALL THE QUESTIONS				
1	Define a) Symmetric Matrix b) Unitary Matrix.	2	1	1,2,3	2
2	Find the rank, index, signature of a quadratic form $2x^2 + 2y^2 + 2z^2 + 2yz$	2	2	1,2,3	3
3	a. Is the matrix $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$ isorthogonal. b. State Cayley-Hamilton theorem.	1 1	1, 2	1,2,3	1 2
	PART-B(5+5+4= 14 Marks)				
	ANSWER ALL THE QUESTIONS				
	(i) Find the rank of the matrix A= $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$ using Echelon form.	5	1	1,2,3	4

	[OR]				
4	(ii) Find whether the following equations are consistent or not, if so solve them $x + 2y + 2z = 2; 3x - 2y - z = 5; \quad 2x - 5y + 3z = -4;$.	5	1	1,2,3	2
5	(i) Find the eigen values and eigen vectors of the following matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$	5	2	1,2,3	2
	[OR]				
	(ii)Verify Cayley-Hamilton theorem for $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$ and hence find $A^{-1}$and A^4.	5	2	1,2,3	3
6	(i)Test the convergence of the series $\frac{3x}{4} + \left(\frac{5}{6}\right)^2 x^2 + \left(\frac{7}{8}\right)^3 x^3 +$	4	3	1,2,3	2
	[OR]				
	(ii) Find the interval of convergence for the following series $\sum(-1)^n \frac{(x+2)}{(2^n+5)}$	4	3	1,2,3,	3

VidyaJyothi Institute of Technology (Autonomous)

(Accredited by NBA Approved By A.I.C.T.E., New Delhi, Permanently Affiliated to JNTU, Hyd)

(Aziz Nagar, C.B.Post, Hyderabad -500075)

Question Paper Title: I B.Tech I Semester Question Paper for Mid Exam II		
Total Duration (H:M):1:30	Course: Matrices and Calculus (Mathematics-I)	Maximum Marks:20
Branch : All Branches	Session : FN	Date: 09 -12-2019

S.No	Course Outcomes
1	Write the matrix representation of system of linear equations and identify the consistency of the system of equations.
2	Find the Eigen values and Eigen vectors of the matrix and discuss the nature of the Quadratic form
3	Analyze the convergence of sequence and series
4	Discuss the applications of mean value theorems to the mathematical problems, Evaluation of improper integrals using Beta and Gamma functions
5	Examine the extrema of functions of two variables with or without constraints.

Blooms level	
Remember	I
Understand	II
Apply	III
Analyze	IV
Evaluate	V
Create	VI

Q.No	Questions	Marks	CO	PO	BL
	PART-A $3\times 2 = 6M$				
	ANSWER ALL THE QUESTIONS				
1	Show that $B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$	2	4	1,2,3,4&11	2
2	If $u=\frac{yz}{x}, v=\frac{zx}{y}, w=\frac{xy}{z}$ *find* $\frac{\partial(u,v,w)}{\partial(x,y,z)}$	2	5	1,2,3,4&11	1,3
3	a. State Cauchy's mean value theorem. b. Show that JJ'=1 for $x=u(1-v), y=uv$	1 1	4 5	1,2,3, 4&11	1,2
	PART-B (5+5+4= 14 Marks)				
	ANSWER ALL THE QUESTIONS				
	(i)(a)Test the convergence of the series $\sum_{n=1}^{\infty}\frac{1\cdot3\cdot5\cdots(2n+1)}{2\cdot5\cdot8\cdots(3n+2)}$ (b)Test for the convergence of	4	3	1,2,3,4&11	2,3

4.	$\frac{2}{1^2}x+\frac{3^2}{2^3}x^2+.....\frac{(n+1)^n}{n^{n+1}}x^n+....,(x>0)$				
	[OR]				
	(ii)(a)Test the series for absolute/conditional convergence $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2+1}$ (b) Examine the convergence of $\frac{1}{5\cdot9\cdot13}-\frac{1}{9\cdot13\cdot17}+\frac{1}{13\cdot17\cdot21}-\cdots\cdots$	4	3	1,2,3,4&11	2,3
5.	(i) (a) Prove that $\int_0^{\pi/2}\sin^2\theta\cos^4\theta\, d\theta=\frac{\pi}{32}$ (b) Prove that $\int_0^1\frac{x^2dx}{\sqrt{1-x^4}}*\int_0^1\frac{dx}{\sqrt{1+x^4}}=\frac{\pi}{4\sqrt{2}}$	5	4	1,2,3,4&11	3
	[OR]				
	(i) If a<b prove that $\frac{b-a}{1+b^2}<\tan^{-1}b-\tan^{-1}a<\frac{b-a}{1+a^2}$ using Lagrange's mean value theorem. Deduce that $\frac{\pi}{4}+\frac{3}{25}<\tan^{-1}\frac{4}{3}<\frac{\pi}{4}+\frac{1}{6}$	5	4	1,2,3,4&11	2
6.	(i) (a) Determine whether the following functions are functionally dependent or not. If functionally dependent then find functional relation $u=xy+yz+zx, v=x^2+y^2+z^2 \,\&\, w=x+y+z$ (b) If $x+y+z=u, y+z=uv, z=uvw, find\ \frac{\partial(x,y,z)}{\partial(u,v,w)}$	5	5	1,2,3,4&11	1,2
	[OR]				
	(ii) Use the method of the Lagrange's multipliers to find volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.	5	5	1,2,3,4&11	1,3

14. ASSIGNMENT (CO, PO, BL)

Vidya Jyothi Institute of Technology

(An Autonomous Institution)

(Accredited by NAAC, Approved by AICTE New Delhi & Permanently Affiliated to JNTUH)

Aziz Nagar Gate, C.B. Post, Hyderabad-500 075

Unit Wise Assignments (With different Levels of thinking – Blooms Taxonomy and Course Outcomes)

	UNIT-I Matrices and Linear System of Equations	**CO's**	**BL**
1	Find the rank of the following matrices by using Echelon form $(a) A=\begin{bmatrix}0&1&-3&-1\\1&0&1&1\\3&1&0&2\\1&1&-2&0\end{bmatrix}, (b) A=\begin{bmatrix}1&4&3&-2&1\\-2&-3&-1&4&3\\-1&6&7&2&9\\-3&3&6&6&12\end{bmatrix},$	1	1
2	Reduce the matrix $A=\begin{bmatrix}2&-2&0&6\\4&2&0&2\\1&-1&0&3\\1&-2&1&2\end{bmatrix}$ to normal form and hence find the rank.	1	1
3	Find whether the following equations are consistent or not, if so solve them $(a) x+y+z=6; 2x+3y-2z=2; 5x+y+2z=13.$ $(b) x+2y+2z=2; 3x-2y-z=5; 2x-5y+3z=-4; x+4y+6z=0.$	1	1
4	Discuss for what values of λ, μ the simultaneous equations $x+y+z=6; x+2y+3z=10; x+2y+\lambda z=\mu$ have (i) no solution (ii) a unique solution (iii) an infinite number of solutions.	1	3
5	Solve the following system of equations $(a) x+3y-2z=0; 2x-y+4z=0; x-11y+14z=0.$ $(b) 4x+2y+z+3w=0; 6x+3y+4z+7w=0; 2x+y+w=0.$ $(c) x+y+w=0; y+z=0; x+y+z+w=0; x+y+2z=0$	1	3
6	Solve the following system of equation by using LU decomposition method $2x+y+z=2; x+3y+2z=2; 3x+y+2z=2.$	1	
	Unit-II: Eigen Values and Eigen Vectors		
1	1. Find the Eigen values and Eigen vectors of the following matrix. $A=\begin{bmatrix}6&-2&2\\-2&3&-1\\2&-1&3\end{bmatrix}$	2	1

2	(a)Prove that if λ is an Eigen value of a non-singular matrix A, then $\frac{\lvert A\rvert}{\lambda}$ is an Eigen value of the matrix adj A. (b)Prove that if λ is an Eigen value of an orthogonal matrix then $\frac{1}{\lambda}$ is also its Eigen value.	2	4
3	Verify Cayley-Hamilton theorem for $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and hence find A^{-1}.	**2**	**3**
4	Diagomalize the matrix $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ and hence find A^4	2	1
5	**Reduce the Quadratic for** form $3x^2 + 5y^2 + 3z^2 - 2xy - 2yz + 2zx$ to the canonical form by orthogonal transformation.	2	3
	Unit-III: Sequences & Series		
1	Test the convergence of the series $\sum \frac{1}{2^n + 3^n}$.	3	3
2	Test the convergence of the series $\sum \frac{1}{n} \log\left(\frac{n+1}{n}\right)$.	3	4
3	Test the convergence of the series $\frac{3x}{4} + \left(\frac{5}{6}\right)^2 x^2 + \left(\frac{7}{8}\right)^3 x^3 +$	3	4
4	Find the interval of convergence for the following series $\sum (-1)^n \frac{(x+2)}{(2^n+5)}$	3	4
5	Find whether the series $\sum (-1)^n \frac{\sin(1/\sqrt{n})}{(n-1)}$ is absolute convergent or conditional convergent.	3	4
	Unit-IV: Beta & Gamma Functions and Mean Value Theorems		
1	(a)Verify Rolle's Theorem for $f(x) = \log\left(\frac{x^2 + ab}{(a+b)x}\right)$ in [a,b]. (b)Verify Lagrange's theorem for $f(x) = (x-1)(x-2)(x-3)$ *on* $[0,4]$ (b)Verify Cauchy's mean value theorem for $f(x) = \sqrt{x}$ *and* $g(x) = \frac{1}{\sqrt{x}}$ *in* $[a,b], 0 < a < b$	4	3
2	(a)If a<b prove that $\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$ using Lagrange's mean value theorem. Deduce the following	4	1

	$\frac{\pi}{4}+\frac{3}{25}<\tan^{-1}\frac{4}{3}<\frac{\pi}{4}+\frac{1}{6}$ (b)Prove that $\frac{\pi}{3}-\frac{1}{5\sqrt{3}}>\cos^{-1}\frac{3}{5}>\frac{\pi}{3}-\frac{1}{8}$ using Lagarng's theorem.		
3	(a)Prove that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ (b)Prove that $\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, where $m>0,\ n>0$	4	3
4	Evaluate a) $\int_0^1 x^4\left(\log\frac{1}{x}\right)^3 dx$ b) $\int_0^\infty a^{-bx^2}dx$ & $\int_0^\infty 3^{-4x^2}dx$ (c) $\int_0^{\pi/2}\sin^2\theta\cos^4\theta\, d\theta$ (d) $\int_0^1 x^{5/2}(1-x^2)^{3/2}dx$	4	3
	Unit-V: Functions of several variables		
1	(a)If $x+y+z=u, y+z=uv, z=uvw, find\ \frac{\partial(x,y,z)}{\partial(u,v,w)}$ (b)If $u=\frac{yz}{x}, v=\frac{zx}{y}, w=\frac{xy}{z}\ find\ \frac{\partial(u,v,w)}{\partial(x,y,z)}$ ©If $u=x^2-y^2, v=2xy\ where\ x=r\cos\theta, y=r\sin\theta\ show\ that\ J(\frac{u,v}{r,\theta})=4r^3$	5	3
2	Determine whether the following functions are functionally dependent or not. If functionally dependent ,find the relation between them $u=x+y+z, v=xy+yz+zx, w=x^2+y^2+z^2$	5	3
3	Find the extreme values of the following functions (a) $u=x^3+3xy^2-3x^2-3y^2+4$ (b) $f(x,y)=x^3+y^3-3axy$	5	3
4	(a)Find the minimum value of $x^2+y^2+z^2$ given that $xyz=a^3$. (b) Find the minimum value of $u=x^2y^3z^4$ if $2x+3y+4z=a$.	5	1
5	A rectangular box open at the top is to have volume of 32 cubic ft. find the dimension of box requiring least material for its construction.	5	

Vidya Jyothi Institute of Technology

(An Autonomous Institution)

(Accredited by NAAC, Approved by AICTE New Delhi & Permanently Affiliated to JNTUH)

Aziz Nagar Gate, C.B. Post, Hyderabad-500 075

S. No.	Name of the Topic	Expected date of Completion	Date of Completion	Teaching Aid/ Teaching Methodology
UNIT – 1: Matrices and Linear System of Equations				
1	Matrices-Introduction	09-08-2019, 13-08-2019	09-08-2019, 13-08-2019	Chalk and Talk
2	Real Matrices-Symmetric, Skew-Symmetric & Orthogonal Matrices	14-08-2019, 16-08-2019, 17-08-2019	14-08-2019, 16-08-2019, 17-08-2019	Chalk and Talk
3	Complex Matrices- Hermitian, Skew-hermitian & Unitary Matrices	19-08-2019	19-08-2019	Chalk and Talk
4	Elementary row transformations-Rank-Echelon form	20-08-2019, 21-08-2019	20-08-2019, 21-08-2019	Chalk and Talk
5	Rank-Normal form	22-08-2019	22-08-2019	Chalk and Talk
6	Solutions of system of linear equations	23-08-2019	23-08-2019	Chalk and Talk
7	Consistency of system of non-homogeneous linear equations	27-08-2019	27-08-2019	Chalk and Talk
8	system of homogeneous linear equations	28-08-2019	28-08-2019	Chalk and Talk
9	Solutions of system of equations by Gauss Jordan method	29-08-2019, 30-08-2019	29-08-2019, 30-08-2019	Chalk and Talk
10	Solutions of system of equations by LU Decomposition method	03-09-2019	03-09-2019	Chalk and Talk
11	Revision			Chalk and Talk
UNIT – 2: Eigen Values and Eigen Vectors				
12	Eigen values and Eigen vectors Introduction	04-09-2019	11-09-2019,	Chalk and Talk
13	Properties	06-09-2019	13-09-2019,	Chalk and Talk Video Presentation
14	Eigen values and Eigen vectors Problems	07-09-2019	16-09-2019,	Chalk and Talk
15	Eigen values and Eigen vectors Problems	11-09-2019,	17-09-2019,	Chalk and Talk Video

				Presentation
16	Cayley-Hamilton theorem(without proof)	13-09-2019	18-09-2019	Chalk and Talk Video Presentation
17	Inverse and powers of a matrix by C-H theorem	16-09-2019	20-09-2019 23-09-2019	Chalk and Talk Video Presentation
18	Digitalization of Matrix	17-09-2019 18-09-2019,	24-09-2019 26-09-2019	Chalk and Talk
19	Quadratic forms	20-09-2019	30-09-2019	Book I and Book II
20	Reduction of Quadratic form to Canonical form by Orthogonal Transformation	23-09-2019 24-09-2019	23-10-2019	Book I and Book II
21	Nature, Index and Signature of Quadratic Form	26-09-2019	25-10-2019	Book I and Book II
22	Revision	30-09-2019	25-10-2019	Book I and Book II
UNIT – 3: Sequences & Series				
23	**Sequences & Series Introduction**	23-10-2019	26-10-2019	Chalk and Talk
24	Basic definitions of Sequences and series	23-10-2019	28-10-2019	Chalk and Talk
25	Convergence and divergence	25-10-2019	29-10-2019	Chalk and Talk
26	Ratio test	26-10-2019 28-10-2019	31-10-2019 01-11-2019	Chalk and Talk
27	Comparison test	29-10-2019	02-11-2019	Chalk and Talk
28	Raabe's test	31-10-2019	04-11-2019	Chalk and Talk
29	Cauchy's root test	01-11-2019	05-11-2019	Chalk and Talk
30	Integral test	02-11-2019	06-11-2019	Chalk and Talk
31	Absolute and conditional convergence	04-11-2019	07-11-2019	Chalk and Talk
32	Revision	05-11-2019	08-11-2019	Chalk and Talk
UNIT – 4: Beta & Gamma Functions and Mean Value Theorems				
33	**Beta & Gamma Functions**	06-11-2019	11-11-2019	Chalk and Talk
34	Gamma and Beta Functions	07-11-2019 08-11-2019	13-11-2019, 14-11-2019	Chalk and Talk
35	Relation between them, their properties	11-11-2019,	15-11-2019,	Chalk and

		13-11-2019	16-11-2019	Talk
36	Evaluation of improper integrals using Gamma / Beta functions	14-11-2019	18-11-2019	Chalk and Talk
37	**Mean Value Theorems**	15-11-2019	18-11-2019	Chalk and Talk
38	Rolle's Theorem	16-11-2019	19-11-2019	Chalk and Talk
39	Lagrange's mean value theorem	18-11-2019	20-11-2019	Chalk and Talk
40	Cauchy's mean value theorem	18-11-2019	20-11-2019	Chalk and Talk
41	Geometrical interpretation of Mean value theorems	15-11-2019 16-11-2019 18-11-2019	16-11-2019 18-11-2019	Chalk and Talk
42	Generalized Mean Value theorem (all theorems without proof)	19-11-2019	21-11-2019	Chalk and Talk
43	Revision	19-11-2019	22-11-2019	Chalk and Talk
UNIT – 5: Functions of several variables				
44	**Functions of several variables**	20-11-2019	23-11-2019	Chalk and Talk
45	Partial Differentiation and total differentiation	20-11-2019	25-11-2019	Chalk and Talk
46	Functional dependence	21-11-2019	25-11-2019	Chalk and Talk
47	Jacobian Determinant	22-11-2019	26-11-2019	Chalk and Talk
48	Maxima and Minima of functions of two variables with constraints and without constraints	23-11-2019, 25-11-2019	27-11-2019	Chalk and Talk
49	Method of Lagrange Multipliers	26-11-2019, 27-11-2019	28-11-2019, 29-11-2019	Chalk and Talk
50	Revision	28-11-2019,	30-11-2019	Chalk and Talk
51	Revision of all units	29-11-2019, 30-11-2019, 02-12-2019, 03-12-2019, 04-12-2019, 05-12-2019	02-12-2019, 03-12-2019, 04-12-2019, 05-12-2019	Chalk and Talk

C. SATHWIK. 19911A05K3 CSE-D MATHS ASSIGNMENT 1

IMPROPER INTEGRALS AND MEAN VALUE THEOREM

1. Show that $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

Sol: We have $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

Put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta \, d\theta$

$\left[\because \frac{d}{dx} x^n = nx^{n-1}\right.$

$\left.\frac{d}{dx}\sin\theta = \cos\theta\right]$

As $x \to 0 \quad \theta \to 0$

$x \to 1 \Rightarrow \sin^2\theta = 1 \Rightarrow \theta = \pi/2$

$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx.$

$B(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \, 2\sin\theta\cos\theta \, d\theta$ $\quad [1-\sin^2\theta = \cos^2\theta]$

$= 2\int_0^{\pi/2} \sin^{2m-2} \cos^{2n-2} \sin\theta\cos\theta \, d\theta$

$= 2\int_0^{\pi/2} \sin^{2m-1} \cos^{2n-1} d\theta$

Let $p = 2m-1 \qquad q = 2n-1$

$p+1 = 2m \qquad q+1 = 2n$

$m = \frac{p+1}{2} \qquad n = \frac{q+1}{2}$

$2\int_0^{\pi/2} \sin^{2m-1} \cos^{2n-1} d\theta = B(m,n)$

$\int_0^{\pi/2} \sin^{2m-1} \cos^{2n-1} d\theta = \frac{1}{2} B(m,n)$

$\int_0^{\pi/2} \sin^{2m-1} \cos^{2n-1} d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

2. Show that $\Gamma 1 = 1$

Sol: We know $\Gamma 1 = \int_0^\infty e^{-x} x^{n-1} dx$

$\Gamma 1 = \int_0^\infty e^{-x} x^0 dx$

$\Gamma 1 = \int_0^\infty e^{-x} dx$

$\Gamma 1 = -\left[e^{-x}\right]_0^\infty$

$\Gamma 1 = -(e^{-\infty} - e^0)$

$\Gamma 1 = -(0-1)$

$\Gamma 1 = 1$

[3] Show that $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1/2 \, \Gamma\frac{p+1}{2} \, \Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}}$

Sol

We have $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \rightarrow [1]$

Put $2m-1 = p \qquad 2n-1 = q$

$2m = p+1 \qquad 2n = q+1$

$m = \frac{p+1}{2} \qquad n = \frac{q+1}{2}$

$\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right) = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$

$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} \beta\left(\frac{p+q}{2}, \frac{q+1}{2}\right) \rightarrow [3]$

We have $\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n} \rightarrow [4]$

from eq (3) & [4]

$\int_0^{\pi/2} \sin^p\theta \cos^q\theta = \frac{1}{2} \frac{\Gamma\frac{p+1}{2} \, \Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}}$ //

[4] Show that $\Gamma 1/2 = \sqrt{\pi}$

Sol <u>1st Method</u>:

We have. $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} \frac{\Gamma\frac{p+1}{2} \, \Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}}$

put $p=0, q=0$

$\int_0^{\pi/2} \sin^0\theta \cos^0\theta \, d\theta = \frac{1}{2} \frac{\Gamma 1/2 \, \Gamma 1/2}{\Gamma 1} \qquad [\because \Gamma 1 = 1]$

$\int_0^{\pi/2} d\theta = \frac{1}{2} [\Gamma 1/2]^2$

$[\theta]_0^{\pi/2} = \frac{1}{2} (\Gamma 1/2)^2$

$\pi/2 - 0 = \frac{1}{2} (\Gamma 1/2)^2$

$\pi = (\Gamma 1/2)^2$

$\Gamma 1/2 = \sqrt{\pi}$

2nd Method :-

We have Beta & Gamma relation $\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$

Put $m = 1/2 \quad n = 1/2$ we get

$$\beta(1/2, 1/2) = \frac{\Gamma 1/2 \; \Gamma 1/2}{\Gamma 1/2 + 1/2}$$

$$\beta(1/2, 1/2) = \frac{(\Gamma 1/2)^2}{\Gamma 1} \qquad [\because \Gamma 1 = 1]$$

$$\beta(1/2, 1/2) = (\Gamma 1/2)^2 \rightarrow \boxed{3}$$

Now we have $\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$

$$\beta(1/2, 1/2) = \int_0^1 x^{-1/2} (1-x)^{-1/2} \, dx$$

Put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta d\theta$

As $x \rightarrow 0 \quad \theta \rightarrow 0$

$x \rightarrow 1 \quad \theta \rightarrow \pi/2$

$$\beta(1/2, 1/2) = \int_0^{\pi/2} (\sin^2\theta)^{-1/2} (1 - \sin^2\theta)^{-1/2} 2\sin\theta\cos\theta d\theta.$$

$$= 2 \int_0^{\pi/2} \sin^{-1}\theta \cos^{-1}\theta \; 2 \sin\theta\cos\theta d\theta$$

$$= 2 \int_0^{\pi/2} d\theta$$

$$= 2 \, [\pi/2 - 0]$$

$$\beta(1/2, 1/2) = 2 \times \pi/2$$

$$= \pi \rightarrow \boxed{4}$$

from eq $\boxed{3}$ & $\boxed{4}$

$$\therefore (\Gamma 1/2)^2 = \pi$$

$$\Gamma 1/2 = \sqrt{\pi}$$

[5] Show that $\int_0^{\pi/2} \sin^n\theta\, d\theta = \int_0^{\pi/2} \cos^n\theta\, d\theta = \frac{1}{2} \cdot \frac{\sqrt{\pi}\,\Gamma\frac{n+1}{2}}{\Gamma\frac{n+2}{2}}$

Sol We have $\int_0^{\pi/2} \sin^p\theta \cos^q\theta = \frac{1}{2} \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}} \rightarrow$ [1]

Put $p = n$ $q = 0$

$$\int_0^{\pi/2} \sin^n\theta \cos^0\theta\, d\theta = \frac{1}{2} \frac{\Gamma\frac{n+1}{2}\,\Gamma\frac{0+1}{2}}{\Gamma\frac{n+0+2}{2}}$$

$$\int_0^{\pi/2} \sin^n\theta = \frac{1}{2} \frac{\Gamma\frac{1}{2}\,\Gamma\frac{n+1}{2}}{\Gamma\frac{n+2}{2}} \qquad [\because \Gamma\tfrac{1}{2} = \sqrt{\pi}]$$

$$\int_0^{\pi/2} \sin^n\theta = \frac{1}{2} \frac{\sqrt{\pi}\,\Gamma\frac{n+1}{2}}{\Gamma\frac{n+2}{2}}$$

Put $p = 0$ $q = n$ in eq [1]

$$\int_0^{\pi/2} \sin^0\theta \cos^n\theta\, d\theta = \frac{1}{2} \frac{\Gamma\frac{0+1}{2}\,\Gamma\frac{n+1}{2}}{\Gamma\frac{n+0+2}{2}}$$

$$\int_0^{\pi/2} \cos^n\theta\, d\theta = \frac{1}{2} \frac{\Gamma\frac{1}{2}\,\Gamma\frac{n+1}{2}}{\Gamma\frac{n+2}{2}} \qquad [\Gamma\tfrac{1}{2} = \sqrt{\pi}]$$

$$\int_0^{\pi/2} \cos^n\theta\, d\theta = \frac{1}{2} \frac{\sqrt{\pi}\,\Gamma\frac{n+1}{2}}{\Gamma\frac{n+2}{2}}$$

Hence

$$\int_0^{\pi/2} \sin^n\theta\, d\theta = \int_0^{\pi/2} \cos^n\theta\, d\theta = \frac{\frac{1}{2}\sqrt{\pi}\,\Gamma\frac{n+1}{2}}{\Gamma\frac{n+2}{2}}$$

[6] Find (i) $\Gamma 9/2$ (ii) $\Gamma -1/2$ (iii) $\Gamma -7/2$ (iv) $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

(i) $\Gamma 9/2$

$\Gamma 9/2 = \Gamma\left(7/2 + 1\right)$ $\qquad \because [\Gamma n+1 = n\Gamma n]$

$= 7/2\, \Gamma 7/2$

$= 7/2 \times \Gamma(5/2 + 1)$ $\qquad [\because \Gamma 1/2 = \sqrt{\pi}$

$= 7/2 \times 5/2\, \Gamma 5/2$

$= \frac{7}{2} \times \frac{5}{2} \times \Gamma(3/2 + 1)$

$= 7/2 \times 5/2 \times 3/2\, \Gamma 3/2$

$= 7/2 \times 5/2 \times 3/2 \times \Gamma(1/2 + 1)$

$= 7/2 \times 5/2 \times 3/2 \times 1/2\, \Gamma 1/2$

$= \frac{105}{16} \sqrt{\pi}$

ii) $\Gamma\left(-\frac{1}{2}\right)$

Sol

$$\Gamma\left(-\tfrac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}}$$

$$= -2\,\Gamma\left(\tfrac{1}{2}\right)$$

$$= -2\sqrt{\pi}$$

$\left[\because n\Gamma n = \Gamma(n+1),\ \Gamma n = \frac{\Gamma(n+1)}{n}\right]$

$\left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$

iii) $\Gamma\left(-\frac{7}{2}\right)$

Sol

$$\Gamma\left(-\tfrac{7}{2}\right) = \frac{\Gamma\left(-\frac{7}{2}+1\right)}{-\frac{7}{2}}$$

$$= -\frac{2}{7} \times \Gamma\left(-\tfrac{5}{2}\right)$$

$$= -\frac{2}{7} \times \frac{\Gamma\left(-\frac{5}{2}+1\right)}{-\frac{5}{2}}$$

$$= -\frac{2}{7} \times \frac{5}{2} \times \Gamma\left(-\tfrac{3}{2}\right)$$

$$\Gamma\left(-\tfrac{7}{2}\right) = -\frac{2}{7} \times -\frac{5}{2} \times \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}}$$

$$= -\frac{2}{7} \times -\frac{5}{2} \times -\frac{2}{3}\,\Gamma\left(-\tfrac{1}{2}\right)$$

$$= -\frac{2}{7} \times -\frac{5}{2} \times -\frac{2}{3} \times \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}}$$

$$= -\frac{2}{7} \times -\frac{2}{5} \times -\frac{2}{3} \times -\frac{2}{1}\,\Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{16}{105}\sqrt{\pi}$$

$\left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$

iv) $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$

Sol We have $\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1}\,dx$ —(1)

put $n = \frac{1}{2}$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-x}\, x^{-1/2}\,dx \rightarrow$$

Put $x = y^2$

$dx = 2y\,dy$

$\left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$

As $x \to 0 \quad y \to 0$

$x \to \infty \quad y \to \infty$

$$\sqrt{\pi} = \int_0^\infty e^{-y^2}\left(y^2\right)^{-1/2} 2y\,dy$$

$\left[\because \int_0^\infty f(y) = \int_0^\infty f(x)\right]$

$$\sqrt{\pi} = 2\int_0^\infty e^{-y^2}\, y^{-1}\, y\,dy$$

$$\frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-y^2}\,dy$$

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

7 Show that if $m>0$, $n>0$

i) $\beta(m+1, n) = \frac{m}{m+n} \beta(m,n)$

ii) $\beta(m, n+1) = \frac{n}{m+n} \beta(m,n)$

iii) $\beta(m, n) = \beta(m+1,n) + \beta(m,n+1)$

Sol (i) We have $\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} \rightarrow (1)$

$$\beta(m+1,n) = \frac{\Gamma m+1 \, \Gamma n}{\Gamma m+n+1}$$

$$= \frac{m \Gamma m \, \Gamma n}{m+n \, \Gamma m+n} \quad \text{from eq } [1]$$

$$= \frac{m}{m+n} \left[\frac{\Gamma m \, \Gamma n}{\Gamma m+n}\right.$$

$$\beta(m+1,n) = \frac{m}{m+n} \beta(m,n) \rightarrow [2]$$

ii) We have $\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$

$$\beta(m, n+1) = \frac{\Gamma m \, \Gamma n+1}{\Gamma m+n+1}$$

$$= \frac{\Gamma m \; n \Gamma n}{m+n \, \Gamma m+1}$$

$$= \frac{n}{m+n} \left[\frac{\Gamma m \, \Gamma n}{\Gamma m+n}\right]$$

$$\beta(m, n+1) = \frac{n}{m+n} \beta(m,n) \rightarrow [3]$$

(iii) given RHS $= \beta(m+1,n) + \beta(m, n+1)$

from eq (3) & [4]

$$\text{RHS} = \frac{m}{m+n} \beta(m,n) + \frac{n}{m+n} \beta(m,n)$$

$$= \beta(m,n) \left[\frac{m+n}{m+n}\right]$$

$$= \beta(m,n)$$

$$= \text{LHS}$$

[8] Evaluate (a) $\int_0^\infty 3^{-4x^2} dx$ (b) $\int_0^\infty x^4 (\log \frac{1}{x})^3 dx$

(a) $\int_0^\infty 3^{-4x^2} dx$

Sol We know that $x = e^{\log x}$

$3 = e^{\log 3}$

$3^{-4x^2} = e^{-4x^2 \log 3}$

put $2x\sqrt{\log 3} = y \Rightarrow dx = \frac{dy}{2\sqrt{\log 3}}$

$y^2 = 4x^2 \log 3$

As $x \to 0$ $y \to 0$

As $x \to \infty$ $y \to \infty$

$\int_0^\infty 3^{-4x^2} dx = \int_0^\infty e^{-y^2} \frac{dy}{2\sqrt{\log 3}} = \frac{1}{2\sqrt{\log 3}} \int_0^\infty e^{-y^2} dy$

Let $y^2 = t$ $\quad y = \sqrt{t}$

$1 dt = 2y dy$ $\quad dy = 1/2 \, t^{-1/2}$

$= \frac{1}{2\sqrt{\log 3}} \int_0^\infty e^{-t} \times 1/2 \, t^{-1/2} dt$

$= \frac{1}{4\sqrt{\log 3}} \int_0^\infty e^{-t} t^{-1/2} dt$ $\quad [\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx]$

$= \frac{1}{4\sqrt{\log 3}} \sqrt{\pi}$

(b) $\int_0^\infty x^4 (\log \frac{1}{x})^3 dx$

Put $\log \frac{1}{x} = t$. $\quad \log x^{-1} = t$

$-1 \log x = t$

$x = e^{-t}$

$dx = -e^{-t} dt$.

As $x \to 0$ $t \to \infty$

$x \to 1$ $t \to 0$

$= \int_\infty^0 (e^{-t})^4 \, t^3 - e^{-t} dt$

$= - \int_\infty^0 e^{-4t} t^3 e^{-t} dt$

$= \int_0^\infty e^{-5t} t^3 dt$

$= \int_0^\infty e^{-5t}\, e^{4-1}\, dt$ $\qquad \int_0^\infty e^{-ky} y^{n-1} dy = \frac{\Gamma n}{k^n}$

$\frac{\Gamma 4}{5^4} = \frac{3!}{5^4} = \frac{6}{5^4}$

[9] Evaluate a) $\int_0^1 x^4 (1-x^2)^{2\,1/3} dx$. b) $\int_0^1 x^4 (8-x^3)^{1/3} dx$.

a) $\int_0^1 x^4 (1-x)^2$

We have $\beta(m,n) = \int_0^\infty x^{m-1} (1-x)^{n-1} dx$

$\beta(m,n) = \int_0^\infty x^{5-1} (1-x)^{3-1} dx$

Here $m=5$ $n=3$

$\beta(5,3) =$

from $\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}$

$\beta(5,3) = \frac{\Gamma 5 \Gamma 3}{\Gamma 5+3}$

$= \frac{4! 2!}{7!}$

$= \frac{1}{105}$

b) $\int_0^2 x^4 (8-x^3)^{1/3} dx$.

Sol We have. Let $x^3 = 8y$ $\quad x = 2y^{1/3}$

$dx = \frac{2}{3} y^{-2/3} dy$

As $x \to 0$ $y \to 0$

$x \to 2$ $y \to 1$.

$\int_0^2 x (8-x^3)^{1/3} dx = \int_0^1 2y^{1/3} (8-8y)^{1/3} \frac{2}{3} y^{-2/3} dy$

$= \frac{8}{3} \int_0^1 y^{1/3} (1-y)^{1/3} y^{-2/3} dy$

$= 8/3 \int_0^1 y^{-1/3+1-1} (1-y)^{1/3+1-1} dy$

$= 8/3\ \beta(2/3, 4/3)$

$\beta \therefore \int_0^2 x(8-x^3)^{1/3}\,dx = \frac{8}{3}\,\beta(2/3, 4/3)$

$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$

$= \frac{8}{3}\,\frac{\Gamma 2/3\,\Gamma 4/3}{\Gamma 2/3+4/3}$

$= \frac{8}{3}\,\frac{\Gamma 2/3\,\Gamma 1/3+1}{\Gamma 2}$

$\Gamma n+1 = n\Gamma n$

$\Gamma n+1 = n!$

$= \frac{8}{3}\,\frac{\Gamma 2/3\,\frac{1}{3}\Gamma 1/3}{1!}$

$= \frac{8}{3}\times\frac{1}{3}\,\Gamma 2/3\,\Gamma 1/3$

$\Gamma n\,\Gamma 1-n = \frac{\pi}{\sin n\pi}$

$= 8/3\times 1/3\,\Gamma 1/3\,\Gamma 1-1/3$

$= \frac{8}{9}\,\frac{\pi}{\sin \pi/3}$

$= \frac{8}{9}\times\frac{\pi\times 2}{\sqrt{3}}$

$= \frac{16\pi}{9\sqrt{3}}$

10\. Show that $\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$.

We have

$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

Put $x = 1/y \quad dx = \frac{-1}{y^2}\,dy$

As $x \to 1 \quad y \to 1$

$x \to \infty \quad y \to 0$

$\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_1^0 \frac{(1/y)^{m-1}}{(1+1/y)^{m+n}}\left(\frac{-1}{y^2}\right)dy$

$= \int_0^1 \frac{y^{-m+1}}{(y+1)^{m+n}}\,y^{m+n}\,y^{-2}\,dy$

$= \int_0^1 \frac{y^{-m+1+m+n-2}}{(y+1)^{m+n}}\,dy$

$$= \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}} dy \qquad \left[\int f(x) = \int f(y)\right]$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx.$$

$$\beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$= \int_0^1 \frac{x^{m+1} + x^{n-1}}{(1+x)^{m+n}} dx$$

[11] Show that $\int_b^a (x-b)^{m-1} (a-x)^{n-1} dx = (a-b)^{m+n-1} \beta(m,n)$

We have $\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx.$

Put $x = \frac{y-b}{a-b} \Rightarrow y = x(a-b)+b$

$dx = \frac{y}{a-b} dy$

As $x \to 0 \Rightarrow y \to b$

$x \to 1 \Rightarrow y \to a.$

then $\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$

$$= \int_b^a \left(\frac{y-b}{a-b}\right)^{m-1} \left(1 - \left(\frac{y-b}{a-b}\right)\right)^{n-1} \cdot \frac{y}{a-b} dy$$

$$= \int_b^a \frac{(y-b)^{m-1}}{(a-b)^{m-1}} \left(\frac{a-y}{a-b}\right)^{n-1} \frac{y}{(a-b)} dy$$

$$= \int_b^a \frac{(y-b)^{m-1} (a-y)^{n-1} y}{(a-b)^{m-1} (a-b)^{n-1} (a-b)} dy$$

$$= \int_b^a \frac{(y-b)^{m-1} (a-y)^{n-1}}{(a-b)^{m-1+n-1+1}} dy$$

$$= \int_b^a \frac{(y-b)^{m-1} (a-y)^{n-1} dy}{(a-b)^{m+n-1}}.$$

$$\beta(m,n) = \frac{1}{(a-b)^{m+n-1}} \int_b^a (y-b)^{m-1} (a-y)^{n-1}\, dy$$

$$= \frac{1}{(a-b)^{m+n-1}} \int_b^a (x-b)^{m-1} (a-x)^{n-1}\, dx$$

$$= (a-b)^{m+n-1} \int_b^a (x-b)^{m-1} (a-x)^{n-1}\, dx$$

$$\int_b^a (x-b)^{m-1} (a-x)^{n-1}\, dx = (a-b)^{m+n-1} \beta(m,n)$$

[12] Show that $\int_{-1}^{1} (1+x)^{m-1} (1-x)^{n-1}\, dx = 2^{m+n-1} \beta(m,n)$

Sol

Let $x = \frac{1+y}{2} \Rightarrow y = 2x-1$

$dx = 1/2\, dy$

As $x \to 0 \Rightarrow y \to -1$

As $x \to 1 \Rightarrow y \to 1$

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx.$$

$$= \int_{-1}^{1} \left(\frac{1+y}{2}\right)^{m-1} \left(1 - \frac{1+y}{2}\right)^{n-1} 1/2\, dy$$

$$= \int_{-1}^{1} \frac{(1+y)^{m-1}}{2^{m-1}} \frac{(1+y)^{n-1}}{2^{n-1}} 1/2\, dy$$

$$= \int_{-1}^{1} \frac{(1+y)^{m-1} (1-y)^{n-1}}{2^{m-1+n-1+1}}\, dy$$

$$= \int_{-1}^{1} \frac{(1+y)^{m-1} (1-y)^{n-1}}{2^{m+n-1}}\, dy$$

$$= \frac{1}{2^{m+n-1}} \int_{-1}^{1} (1+y)^{m-1} (1-y)^{n-1}\, dy$$

$$= \frac{1}{2^{m+n-1}} \int_{-1}^{1} (1+x)^{m-1} (1-x)^{n-1}\, dx.$$

$$\int_0^1 (1+x)^{m-1} (1-x)^{n-1}\, dx = 2^{m+n-1} \beta(m,n)$$

[13] show that $\frac{\beta(p,q+1)}{q} = \frac{\beta(p+1,q)}{p} = \frac{\beta(p,q)}{p+q}$ where $p,q>0$

Sol:

$$\frac{\beta(p,q+1)}{q} = 1/q\ \beta(p,q+1) = 1/q \int_0^1 x^{p-1}(1-x)^{q-1+1}dx$$

$$= 1/q \int_0^1 x^{p-1}(1-x)^q dx.$$

$$= \frac{1}{q}\left[(1-x)^q \int_0^1 x^{p-1}dx - \int\left(\frac{d}{dx}(1-x)^q \int x^{p-1}\right)dx\right]$$

$$= \frac{1}{q}\left[\left[(1-x)^q \frac{x^p}{p}\right]_0^1 - \int_0^1 -q(1-x)^{q-1}\frac{x^p}{p}dx\right]$$

$$= \frac{1}{q}\left[[0-0] + \frac{q}{p}\int_0^1 (1-x)^{q-1} x^p dx\right]$$

$$= \frac{1}{q}\left[+\frac{q}{p}\int_0^1 x^p(1-x)^{q-1}dx\right]$$

$$= \frac{1}{p}\beta(p+1,q)$$

$$= \frac{\beta(p+1,q)}{p}$$

$$\frac{\beta(p,q+1)}{q} = \frac{\beta(p+1,q)}{p}$$

$$= \frac{1}{p}\beta(p+1,q)$$

$$= \frac{1}{p}\int_0^1 x^p(1-x)^{q-1}dx$$

$$= \frac{1}{p}\int_0^1 x^{p-1}(1-(1-x))(1-x)^{q-1}dx.$$

$$= \frac{1}{p}\left[\int_0^1 x^{p-1}(1-x)^{q-1} - \int_0^1 x^{p-1}(1-x)^q dx\right]$$

$$= \frac{1}{p}[\beta(p,q) - \beta(p,q+1)]$$

$$\frac{\beta(p,q+1)}{q} = \frac{\beta(p,q)}{p} - \frac{\beta(p,q+1)}{p}$$

$$\frac{\beta(p,q+1)}{q} + \frac{\beta(p,q+1)}{p} = \frac{\beta(p,q)}{p}$$

$$\beta(p,q+1)\left[\frac{p+q}{pq}\right] = \frac{\beta(p,q)}{p}$$

$$\frac{\beta(p,q+1)}{q} = \frac{\beta(p,q)}{p+q}.$$

[15] Show that $\int_0^2 [(1-x^2)]^{-1/n} dx = \frac{1}{n} \frac{(\Gamma\frac{1}{n})^2}{2\Gamma\frac{2}{n}}$

Sol Take $\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}$

Put $x^n = t \Rightarrow x = t^{1/n}$

$n x^{n-1} dx = dt \quad dx = \frac{1}{n} t^{1/n-1} dt$

As $x \to 0 \Rightarrow t \to 0$

$x \to 1 \Rightarrow t \to 1$

$\int_0^2 (1-x^n)^{-1/n} dx = \int_0^1 (1-t)^{-1/n} dx$

$= \int_0^1 (1-t)^{-1/n} \times \frac{1}{n} t^{1/n-1} dt$

$= \frac{1}{n} \int_0^1 (1-t)^{1/n} \frac{t^{1/n}}{t} dt$

$= \frac{1}{n} \int_0^1 t^{\frac{1}{n}-1} (1-t)^{1/n+1-1} dt$

$= \frac{1}{n} B(\frac{1}{n}, \frac{1}{n}+1)$

$= \frac{1}{n} \frac{\Gamma 1/n \, \Gamma 1/n+1}{\Gamma 1/n + 1/n + 1}$

$= \frac{1}{n} \frac{\Gamma\frac{1}{n} \, 1/n \, \Gamma 1/n}{\Gamma\frac{2}{n}+1}$

$= \frac{1}{n^2} \frac{[\Gamma 1/n]^2}{\frac{2}{n} \Gamma\frac{2}{n}}$

$\int_0^2 (1-x^n)^{1/n} dx = \frac{1}{n} \frac{[\Gamma 1/n]^2}{2\Gamma\frac{2}{n}}$

[16] Evaluate $\int_0^1 \frac{dx}{\sqrt{-\log x}}$

Sol Put $-\log x = t$

$x = e^{-t}$

$dx = -e^{-t} dt$

As $x \to 0 \Rightarrow t \to \infty$

$x \to 1 \to t \to 0$

$$\int_0^1 \frac{dx}{\sqrt{-\log x}} = \int_\infty^0 \frac{1}{\sqrt{t}} \cdot -e^{-t} dt$$

$$= \int_0^\infty \frac{e^{-t}}{\sqrt{t}} dt$$

$$= \int_0^\infty e^{-t} t^{-1/2} dt$$

$$= \int_0^\infty e^{-t} t^{1/2} dt$$

$$= \int_0^\infty e^{-t} t^{1/2} dt \qquad \Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

$$\Gamma n = \Gamma 1/2 = \sqrt{\pi}$$

[17] Prove that a) $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma 1/n}{\Gamma 1/n + 1/2}$.

b) $\int_0^1 \frac{dx}{(1-x^n)^{1/n}} = \frac{\pi}{n} \operatorname{cosec}\left(\frac{\pi}{n}\right)$

a) $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma 1/n}{\Gamma 1/n + 1/2}$.

Put $x^n = t$

$x = t^{1/n}$

$dx = 1/n \, t^{1/n - 1} dt$

As $x \to 0 \quad t \to 0$

$x \to 1 \quad t \to 1$

$$\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \int_0^1 \frac{1}{\sqrt{1-t}} \, 1/n \, t^{1/n - 1}$$

$$= \frac{1}{n} \int_0^1 \frac{t^{1/n}}{t\sqrt{1-t}} dt$$

$$= 1/n \int_0^1 \frac{t^{1/n - 1}}{(1-t)^{1/2}} dt$$

$$= \frac{1}{n} \int_0^1 t^{1/n - 1} (1-t)^{-1/2} dt$$

$$B(m,n) = 1/n \int_0^1 x^{m-1} (1-x)^{n-1} dx.$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{1}{n}\int_0^1 t^{1/n-1} (1-t)^{1/2-1} dt$$

$$= 1/n \; \beta(1/n, 1/2)$$

$$= 1/n \frac{\Gamma 1/n \, \Gamma 1/2}{\Gamma \frac{1}{n}+1/2}$$

$$= \frac{\sqrt{\pi}}{n} \frac{\Gamma 1/n}{\Gamma 1/n+1/2}$$

(b) $\int_0^1 \frac{dx}{(1-x^n)^{1/n}} = \frac{\pi}{n} \operatorname{cosec} \frac{\pi}{n}$

Sol

Put $x^n = \sin^2\theta$

$x = (\sin\theta)^{2/n}$

$dx = \frac{2}{n} (\sin\theta)^{\frac{2}{n}-1} \cos\theta \, d\theta$

As $x \to 0$
$\theta \to 0$
$x \to \pm 1$
$\theta \to \pi/2$

$$\Rightarrow \int_0^{\pi/2} \frac{2}{n} \frac{1}{(1-\sin^2\theta)^{1/n}} \frac{2}{n}(\sin\theta)^{\frac{2\theta}{n}-1} \cos\theta \, d\theta$$

$$= \frac{2}{n} \int_0^{\pi/2} \frac{1}{(\cos^2\theta)^{1/n}} \frac{(\sin\theta)^{2/n}}{\sin\theta} \cdot \cos\theta \, d\theta$$

$$= \frac{2}{n} \int_0^{\pi/2} \frac{1}{\cos^{2/n}\theta} \frac{(\sin\theta)^{2/n}}{\sin\theta} \cos\theta \, d\theta$$

$$= \frac{2}{n} \int_0^{\pi/2} \cos^{1-2/n}\theta \, \sin^{2/n-1}\theta \, d\theta$$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta = 1/2 \, \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{2}{n} \times 1/2 \, \beta\left(\frac{\frac{2}{n}-1+1}{2}, \frac{1-2/n+1}{2}\right)$$

$$= \frac{1}{n} \beta\left(\frac{\frac{2}{n}}{2}, \frac{2-2/n}{2}\right)$$

$$= \frac{1}{n} \beta\left(1/n, 1-1/n\right)$$

$$= \frac{1}{n} \frac{\Gamma 1/n \, \Gamma 1-1/n}{\Gamma \frac{1}{n}+1-1/n}$$

$= \frac{1}{n} \cdot \frac{\pi}{\sin(1/n)\pi}$ $\qquad \Gamma n \, \Gamma 1 - 1/n = \frac{\pi}{\sin n\pi}$

$= \frac{\pi}{n} \operatorname{cosec}(\frac{1}{n})$ //

[14] Show that $\int_0^\infty x^4 e^{-x^2} dx = \frac{3\sqrt{\pi}}{8}$

Put $x^2 = t$

$x = t^{1/2}$

$dx = 1/2 \, t^{-1/2} dt$

As $x \to 0 \Rightarrow t \to 0$

$x \to \infty \Rightarrow t \to \infty$

$\int_0^\infty x^4 e^{-x^2} dx = \int_0^\infty t^2 e^{-t} \, 1/2 \, t^{-1/2} dt$

$= 1/2 \int_0^\infty e^{-t} t^{3/2} dt$

$= 1/2 \int_0^\infty e^{-t} t^{5/2 - 1} dt$

$\Gamma n = \int_0^\infty e^{-ky} \cdot x^{n-1} dx = \frac{\Gamma n}{k^n}$

$= 1/2 \cdot \Gamma 5/2$

$= 1/2 \, \Gamma 3/2 + 1$

$= 1/2 \; 3/2 \; \Gamma 3/2$

$= 1/2 \times 3/2 \; \Gamma 1/2 + 1$

$= 1/2 \times 3/2 \times 1/2 \; \Gamma 1/2$

$= \frac{3}{8} \Gamma 1/2$

$= \frac{3\sqrt{\pi}}{8}$ //

[18] Prove that $\int_0^1 x^m (\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$ where n is positive integer

Sol

Let $I = \int_0^1 x^m (\log x)^n dx$

Put $\log x = -t$

$x = e^{-t}$

$dx = -te^{-t} dt$

As $x \to 0 \Rightarrow t \to \infty$

$x \to 1 \to t \to 0$

$$\int_0^1 x^m (\log x)^n dx = \int_\infty^0 (-t)^n e^{-t} dt)\, e^{-mt}$$

$$= (-1)^n \int_\infty^0 t^n e^{-t} e^{-mt} dt$$

$$= (-1)^n \int_0^\infty t^n e^{-t(m+1)} dt$$

$$= (-1)^n \int_0^\infty t^{n+1-1} e^{-t(m+1)} dt$$

$$\int_0^\infty e^{-ky} y^{x-1} dy = \frac{\Gamma n}{k^n}$$

$$= (-1)^n \frac{\Gamma n+1}{(m+1)^{n+1}}$$

$$= \frac{(-1)^n n!}{(m+1)^{n+1}}$$

[19] Find the value $\int_0^{\pi/2} \sin^5\theta \cos^{7/2}\theta \, d\theta$

Sol

We know that $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$$\int_0^{\pi/2} \sin^5\theta \cos^{7/2}\theta \, d\theta = \frac{1}{2} \beta\left(\frac{5+1}{2}, \frac{7/2+1}{2}\right)$$

$$= \frac{1}{2} \beta\left(6/2, \frac{9}{4}\right)$$

$$= \frac{1}{2} \beta(3, 9/4)$$

We have $\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$

$$= \frac{1}{2} \times \frac{\Gamma 3 \, \Gamma \frac{9}{4}}{\Gamma 3+9/4}$$

$$= \frac{2!}{2} \cdot \frac{\Gamma \frac{9}{4}}{\Gamma \frac{21}{4}}$$

$$= \frac{\Gamma \frac{9}{4}}{\Gamma 1+\frac{17}{4}} = \frac{\Gamma \frac{9}{4}}{\frac{17}{4} \Gamma \frac{17}{4}} = \frac{\Gamma \frac{9}{4}}{\frac{17}{4} \Gamma 1+\frac{13}{4}} = \frac{\Gamma \frac{9}{4}}{\frac{17}{4} \frac{13}{4} \times \Gamma \frac{13}{4}}$$

$$= \frac{\Gamma \frac{9}{4}}{\frac{13}{4} \times \frac{17}{4} \Gamma 1+\frac{9}{4}} = \frac{\Gamma 9/4}{\frac{13}{4} \times \frac{17}{4} \times \frac{9}{4} \Gamma 9/4} = \frac{64}{13 \times 17 \times 9}$$

[20] Find the value of $\int_0^{\pi/2} \sin^7\theta \, d\theta$

Sol: We know that $\int_0^{\pi/2} \sin^n\theta \, d\theta = \frac{\frac{\sqrt{\pi}}{2} \Gamma \frac{n+1}{2}}{\Gamma \frac{n+2}{2}}$

$n = 7$

$$\int_0^{\pi/2} \sin^7\theta \, d\theta = \frac{\frac{\sqrt{\pi}}{2} \Gamma \frac{7+1}{2}}{\Gamma \frac{7+2}{2}} = \frac{\frac{\sqrt{\pi}}{2} \Gamma 4}{\Gamma 9/2}$$

$$= \frac{\frac{\sqrt{\pi}}{2} \times 3!}{9/2 \times 7/2 \times 5/2 \times 3/2 \times 1/2 \times \sqrt{\pi}} = \frac{3}{\frac{105}{16}} = \frac{48}{105}$$

[21] Find the value of $\int_0^{\pi/2} \cos^{11}\theta \, d\theta$

We know that $\int_0^{\pi/2} \cos^n\theta \, d\theta = \frac{\frac{\sqrt{\pi}}{2} \Gamma \frac{n+1}{2}}{\Gamma \frac{n+1}{2}}$

$n = 11$

$$\int_0^{\pi/2} \cos^{11}\theta \, d\theta = \frac{\frac{\sqrt{\pi}}{2} \Gamma \frac{11+1}{2}}{\Gamma 11+1/2} = \frac{\sqrt{\pi}}{2} \times \frac{\Gamma 6}{\Gamma 13/2} = \frac{\sqrt{\pi}}{2} \times \frac{5!}{\frac{11}{2} \times 9/2 \times 7/2 \times 5/2 \times 3/2 \times 1/2 \times \sqrt{\pi}}$$

$$= \frac{120 \times 64}{11 \times 9 \times 7 \times 5 \times 3} = \frac{7{,}680}{10{,}395}$$

22 Show that $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi$

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \int_0^{\pi/2} \sin^{-1/2}\theta\, d\theta$$

$$\int_0^{\pi/2} \sin^n\theta\, d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma\frac{n+1}{2}}{\Gamma\frac{n+2}{2}}$$

$$= \frac{\sqrt{\pi}}{2} \frac{\Gamma 1/4}{\Gamma 3/4}$$

$$\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta\, d\theta$$

$$= \frac{\sqrt{\pi}}{2} \times \frac{\Gamma 3/4}{\Gamma 5/4}$$

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma 1/4}{\Gamma 3/4} \times \frac{\sqrt{\pi}}{2} \frac{\Gamma 3/4}{\Gamma 5/4}$$

$$= \frac{\pi}{4} \times \frac{\Gamma 1/4}{\Gamma 1+1/4}$$

$$= \frac{\pi}{4} \times \frac{\Gamma 1/4}{1/4\, \Gamma 1/4}$$

$$= \frac{4\pi}{4}$$

$$= \pi$$

Hence proved.

23 Show that $\int_0^{\pi/2} (\sqrt{\tan\theta}\, d\theta + \sqrt{\sec\theta})\, d\theta$

Sol:

$$\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta + \int_0^{\pi/2} \sqrt{\sec\theta}\, d\theta$$

$$\int_0^{\pi/2} \frac{\sin^{1/2}\theta}{\cos^{1/2}\theta} d\theta + \int_0^{\pi/2} \frac{1}{\cos^{1/2}\theta} d\theta$$

$$\int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta\, d\theta + \int_0^{\pi/2} \cos^{-1/2}\theta\, d\theta$$

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta\, d\theta = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \qquad \int_0^{\pi/2} \cos^n\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma\frac{n+1}{2}}{\Gamma\frac{n+2}{2}}$$

$= \frac{1}{2} \beta\left(\frac{\frac{1}{2}+1}{2}, \frac{1-\frac{1}{2}}{2}\right) + \frac{\sqrt{\pi}}{2} \frac{\Gamma\frac{1-\frac{1}{2}}{2}}{\Gamma\frac{2-\frac{1}{2}}{2}}$

$= \frac{1}{2} \times \beta\left(\frac{3}{4}, \frac{1}{4}\right) + \frac{\sqrt{\pi}}{2} \frac{\Gamma\frac{1}{4}}{\Gamma\frac{3}{4}}$

$= \frac{1}{2} \times \frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}}{\Gamma 1} + \frac{\sqrt{\pi}}{2} \frac{\Gamma\frac{1}{4}}{\Gamma\frac{3}{4}}$

$= \frac{1}{2} \times \Gamma\frac{1}{4}\,\Gamma 1-\frac{1}{4} + \frac{\sqrt{\pi}}{2} \cdot \Gamma\frac{1}{2}$

$= \frac{1}{2} \times \frac{\pi}{\sin\frac{\pi}{4}} + \frac{\sqrt{\pi}}{2} \Gamma\pi$

[24] Find the value of $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$

Sol given that $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$ $\qquad \int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$\Rightarrow \int_0^{\pi/2} \cos^{1/2} \sin^{-1/2} d\theta$

$p = -\frac{1}{2} \quad q = \frac{1}{2}$

$= \frac{1}{2} \beta\left(\frac{1-\frac{1}{2}}{2}, \frac{1+\frac{1}{2}}{2}\right)$

$= \frac{1}{2} \beta\left(\frac{1}{4}, \frac{3}{4}\right)$

$= \frac{1}{2} \cdot \frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{\Gamma 1}$

$= \frac{1}{2} \Gamma\frac{1}{4}\,\Gamma 1-\frac{1}{4}$

$= \frac{1}{2} \times \frac{\pi}{\sin\frac{\pi}{4}} = \frac{1}{2} \frac{\pi}{1/\sqrt{2}} = \frac{\pi}{\sqrt{2}}$

15. PREVIOUS END SEMESTER QUESTION PAPERS

Vidya Jyothi Institute of Technology (Autonomous)

(Accrediated by NAAC & NBA, Approved by A.I.C.T.E., New Delhi, Permanently Affiliated to JNTU, Hyderabad)
(Aziz Nagar, C.B. Post, Hyderabad-500075)

R19

Subject Code:A31002

B. Tech. I YEAR I SEMESTER REGULAR EXAMINATION, DECEMBER-2019

SUBJECT: Mathematics -I (Matrices & Calculus) **BRANCH:** CE, EEE, MECH & ECE

Time: 3 Hours **Max. Marks: 75**

Note: This question paper contains two ***Parts A and B.***

Part A is compulsory which carries 25 Marks. Answer all the questions.

Part B consists of 5 Questions. Answer all the questions.

Bloom's Level:

Remember	L1	Apply	L3	Evaluate	L5
Understand	L2	Analyze	L4	Create	L6

PART-A ANSWER ALL THE QUESTIONS		Bloom's Level	25 Marks
1	Define Hermitian Matrix and Skew-Hermitian Matrix.	L1	2M
2	Find the value of k such that the rank of $A=\begin{bmatrix}1 & 1 & -1 & 1\\ 1 & -1 & k & -1\\ 3 & 1 & 0 & 1\end{bmatrix}$ is 2	L1	3M
3	If $A=\begin{bmatrix}1 & 3 & 4\\ 0 & 2 & 5\\ 0 & 0 & 3\end{bmatrix}$ Find the Eigen values of $adjA$.	L5	2M
4	Identify the nature of the quadratic form X^TAX if $A=\begin{bmatrix}1 & 1 & 3\\ 1 & 5 & 1\\ 3 & 1 & 1\end{bmatrix}$	L1,L4	3M
5	Test the convergence of the series $\sum\left(1+\frac{1}{n}\right)^{-n^2}$	L4,L6	2M
6	State Raabe's test.	L1	3M
7	Find the value of $\Gamma\left(-\frac{1}{2}\right)$	L3	2M
8	Discuss the applicability of Rolle's theorem $2x^3+x^2-4x-2$ in $\left[-\sqrt{3},\sqrt{3}\right]$	L3	3M
9	If $u=x^2-y^2, v=2xy$, *find* $\frac{\partial(u,v)}{\partial(x,y)}$	L1,L4	2M
10	Find the stationary points of $u=x^3+3x^2+y^2+4xy$	L1	3M
PART-B **ANSWER ALL THE QUESTIONS**		**Bloom's Level**	**50 Marks**
11. i.a)	Find the rank of the matrix by reducing it to normal form $A=\begin{bmatrix}1 & 2 & -1 & 3\\ 4 & 1 & 2 & 1\\ 3 & -1 & 1 & 2\\ 1 & 2 & 0 & 1\end{bmatrix}$	L1,L4	5M

P.T.O

b)	Find whether the following system of equations are consistent. If so, solve them. $x+2y+2z=2; 3x-2y-z=5; 2x-5y+3z=-4; x+4y+6z=0$	L1,L4	5M
	(OR)		
ii.	Solve the following system of equation by using LU decomposition method $2x-3y+10z=3; -x+4y+2z=20; 5x+2y+z=-12$.	L3, L6	10M
12. i.	Find the Eigen values and Eigen vectors of $A=\begin{bmatrix}5 & -2 & 0\\ -2 & 6 & 2\\ 0 & 2 & 7\end{bmatrix}$	L5	10M
	(OR)		
ii.	Reduce the Quadratic form to the canonical form by orthogonal transformation. $3x^2+5y^2+3z^2-2yz+2zx-2xy$ and hence find the rank, index, signature and nature of the quadratic form.	L6,L2	10M
13. i.	Test for the convergence of the series $\sum_{n=1}^{\infty}\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots2n}x^n$	L1	10M
	(OR)		
ii.	Test the series for absolute or conditional convergence $\frac{2}{3}-\frac{3}{4}\cdot\frac{1}{2}+\frac{4}{5}\cdot\frac{1}{3}-\frac{5}{6}\cdot\frac{1}{4}+\ldots\ldots\ldots$	L4, L6	10M
14. i. a)	Evaluate $\int_0^2 x\left(8-x^3\right)^{1/3}dx$ using $\beta-\Gamma$ functions.	L5	5M
b)	Prove that $\int_0^1 x^m(\log x)^n dx=\frac{(-1)^n n!}{(m+1)^{n+1}}$ where n is a positive integer and m>-1.	L5	5M
	(OR)		
ii.	Using the Lagrange's mean value theorem, Show that $\frac{\pi}{3}-\frac{1}{5\sqrt{3}}>\cos^{-1}\frac{3}{5}>\frac{\pi}{3}-\frac{1}{8}$	L1	10M
15. i a)	Verify if $u=2x-y+3z, v=2x-y-z, w=2x-y+z$ are functionally dependent and if so, find a relation between them.	L3	5M
b)	Find the maximum and minimum values of $x^3+3xy^2-15x^2-15y^2+72x$.	L3	5M
	(OR)		
ii.	Use the method of the Lagrange's multipliers to find volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	L1,L4	10M

16. COURSE END SURVEY

Vidya Jyothi Institute of Technology

(An Autonomous Institution)

(Accredited by NAAC, Approved by AICTE New Delhi & Permanently Affiliated to JNTUH)

Aziz Nagar Gate, C.B. Post, Hyderabad-500 075

Department of Humanities & Sciences

Course End Survey Form

Academic year: 2019-20

Regulations:

Name of the student	S. Chakra pani	**Year & Sem**	I year I Sem
Roll number	19911A05N8	**Subject**	Mathematics-I

Dear Student,

We need your help in evaluating the courses offered, by responding the short survey below.

Your feedback is very valuable for us in order to continually improve our program. The aim of this survey is to evaluate how well each of the courses has prepared you to have necessary skills.

Your responses will be kept confidential and will not be revealed to anyone outside the department without your permission.

Please indicate (√) the level to which you agree with the following criterion:

(1: Low 2: Moderate 3: High)

Name of The Course: Mathematics I		**RATING**		
After completing this course the student must demonstrate the knowledge and ability to		3	2	1
CO 1	Write the matrix representation of system of linear equations and identify the consistency of the system of equations.	✓		
CO 2	Find the Eigen values and Eigen vectors of the matrix and discuss the nature of the quadratic form.	✓		
CO 3	Analyse the convergence of sequence and series.	✓		
CO 4	Discuss the applications of mean value theorems to the mathematical problems, Evaluation of improper integrals using Beta and Gamma functions.	✓		
CO 5	Examine the extrima of functions of two variables with/ without constraints.	✓		

Any other comments / suggestions:

S. Chakrapani

Signature

Vidya Jyothi Institute of Technology

(An Autonomous Institution)
(Accredited by NAAC, Approved by AICTE New Delhi & Permanently Affiliated to JNTUH)
Aziz Nagar Gate, C.B. Post, Hyderabad-500 075

Department of Humanities & Sciences

Course End Survey Form

Academic year: 2019-2020
Regulations:

Name of the student	P. Gopi Venkata Ashok	**Year & Sem**	Ist year, Ist sem
Roll number	19911A05N1	**Subject**	Mathematics -I

Dear Student,

We need your help in evaluating the courses offered, by responding the short survey below.

Your feedback is very valuable for us in order to continually improve our program. The aim of this survey is to evaluate how well each of the courses has prepared you to have necessary skills.

Your responses will be kept confidential and will not be revealed to anyone outside the department without your permission.

Please indicate (√) the level to which you agree with the following criterion:
(1: Low 2: Moderate 3: High)

Name of The Course: Mathematics I		**RATING**		
After completing this course the student must demonstrate the knowledge and ability to		3	2	1
CO 1	Write the matrix representation of system of linear equations and identify the consistency of the system of equations.	✓		
CO 2	Find the Eigen values and Eigen vectors of the matrix and discuss the nature of the quadratic form.	✓		
CO 3	Analyse the convergence of sequence and series.	✓		
CO 4	Discuss the applications of mean value theorems to the mathematical problems, Evaluation of improper integrals using Beta and Gamma functions.	✓		
CO 5	Examine the extrima of functions of two variables with/ without constraints.	✓		

Any other comments / suggestions:

____

____

____

Signature

Vidya Jyothi Institute of Technology

(An Autonomous Institution)

(Accredited by NAAC, Approved by AICTE New Delhi & Permanently Affiliated to JNTUH)

Aziz Nagar Gate, C.B. Post, Hyderabad-500 075

Department of Humanities & Sciences

Course End Survey Form

Academic year: 2019 - 20

Regulations:

Name of the student	D. UMA MAHESHWAR RAO	**Year & Sem**	I[st] year I[st] sem
Roll number	19911A05156	**Subject**	M - I

Dear Student,

We need your help in evaluating the courses offered, by responding the short survey below.

Your feedback is very valuable for us in order to continually improve our program. The aim of this survey is to evaluate how well each of the courses has prepared you to have necessary skills.

Your responses will be kept confidential and will not be revealed to anyone outside the department without your permission.

Please indicate (√) the level to which you agree with the following criterion:

(1: Low 2: Moderate 3: High)

Name of The Course: Mathematics I		**RATING**		
After completing this course the student must demonstrate the knowledge and ability to		3	2	1
CO 1	Write the matrix representation of system of linear equations and identify the consistency of the system of equations.	✓		
CO 2	Find the Eigen values and Eigen vectors of the matrix and discuss the nature of the quadratic form.	✓		
CO 3	Analyse the convergence of sequence and series.	✓		
CO 4	Discuss the applications of mean value theorems to the mathematical problems, Evaluation of improper integrals using Beta and Gamma functions.	✓		
CO 5	Examine the extrima of functions of two variables with/ without constraints.	✓		

Any other comments / suggestions:

D. Uma mahesh

Signature

17. CONTENT BEYOND SYLLABUS MAPPING WITH PO'S AND PSO'S

Vidya Jyothi Institute of Technology

(An Autonomous Institution)

(Accredited by NAAC, Approved by AICTE New Delhi & Permanently Affiliated to JNTUH)

Aziz Nagar Gate, C.B. Post, Hyderabad-500 075

Topics beyond Syllabus

S. No	Name of the Topic	
1	Applications of Matrices	PPT
2	Taylor's and McLaren's Series	Material
3		

Regulation : R18
Academic Year : 2019-2020
Program : B. Tech
Year/Sem : I/I sem
Course Name : Matrices and Calculus (Mathematics I)
Course Code : A31002
Contact Hours : 5Lectures/1Tutorial/4Credits
No. of Students : 45

No. of lecture classes taken	63
No. of tutorial classes taken	5
Course delivery modes	Lectures, Demonstration
Technology utilization	Power Point / OHP Slides
Assessment Tools	Internal Mid Examinations, Assignments, End Exam

OVERALL ATTAINMENT (80% DIRECT + 20% INDIRECT)	
DIRECT	3.0
INDIRECT	2.96
OVERALL ATTAINMENT	2.992

APPLICATIONS OF MATRICES

Department of Mathematics
VJIT

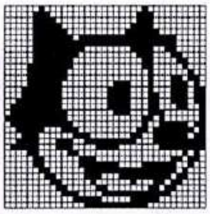
APPLICATIONS OF MATRICES

- Image Processing
- Criptography
- Strength of materials
- Computer Aided Designing (CAD)
- Robotics Engineering
- Finite Element Analysis (FEA) and Finite Element Methods (FEM)
- MATLAB
- Electrical circuits (Structural Matrix)
- Wireless communication and signal processing

IMAGE PROCESSING

Grayscale

Felix the Cat



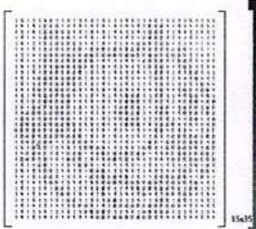


IMAGE PROCESSING....

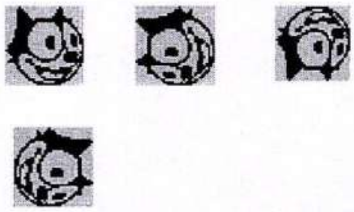


IMAGE PROCESSING....

color images

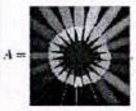


IMAGE PROCESSING....

Color images, in turn, can be represented by three matrices.

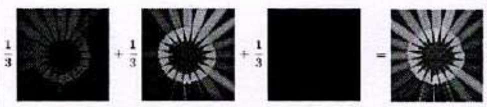
Red, Green and Blue that makes up the image

This color system is known as RGB[3]

The elements of these matrices are numbers between 0 and 255.

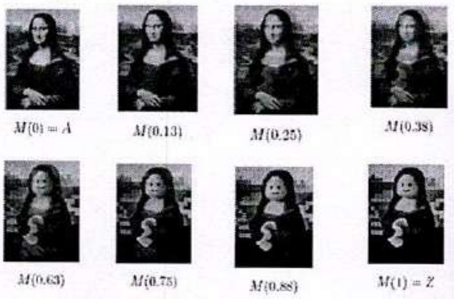
it is possible to represent

$256^3 = 2^{24} = 16777216$ different colors.



USING MATRIX OPERATIONS

$$M(t) = (1-t)A + tZ.$$



NOISE REDUCTION

- The median filter technique is used to remove the noise or reduce their effects, for each element of the matrix that represents the image, we observe its neighboring elements and, then, we arrange them in an ordered list. The median filter consists of choosing the central element of this list and replace the element at the center by this one

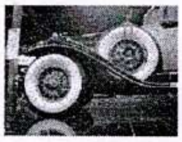
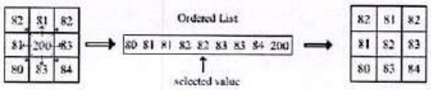


Image with noise

NOISE REDUCTION....



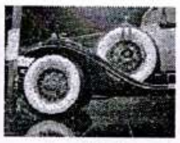


Image with noise

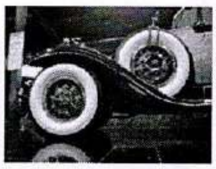


Image with the median filter

CRYPTOGRAPHY

- Cryptography is the technique to encrypting data so that only the relevant person can get the data and relate information.
- This encrypting is done by using an invertible key.
- By using inverse of invertible key the data can be encrypted.
- This process is done using matrices.
- A digital audio or video signal is firstly taken as a sequence of numbers representing the variation over time of air pressure of an acoustic audio signal.
- The filtering techniques are used which depends on matrix multiplication.

CRYPTOGRAPHY....

- "Do Not Worry"
- The message is converted into a sequence of numbers from 1 to 26.For space use digit 0.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

The message "DO NOT WORRY" can be encoded as sequence of numbers

4 15 0 14 15 20 0 23 15 18 18 25

$$\begin{bmatrix} 4 & 15 \\ 0 & 14 \\ 15 & 20 \\ 0 & 23 \\ 15 & 18 \\ 18 & 25 \end{bmatrix}$$

CRYPTOGRAPHY....

- To encrypt this data invertible matrix is used; choose a matrix whose determinant in non-zero.
- Suppose we use an invertible matrix

$$B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

$$\text{Then } X = AB = \begin{bmatrix} 4 & 15 \\ 0 & 14 \\ 15 & 20 \\ 0 & 23 \\ 15 & 18 \\ 18 & 25 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 53 & 64 \\ 42 & 56 \\ 90 & 95 \\ 69 & 92 \\ 84 & 89 \\ 111 & 118 \end{bmatrix}$$

CRYPTOGRAPHY....

- Now the message that will pass in air to the other person is

  53 64 42 56 90 95 69 92 84 89 111 118.
- To unencrypt data first we have to find B^{-1}

$$B^{-1} = \begin{bmatrix} 0.8 & -0.2 \\ -0.6 & 0.4 \end{bmatrix}$$

To get original message we operate B^{-1} on AB =X

$$(AB)B^{-1} = XB^{-1}$$
$$A(BB^{-1}) = XB^{-1}$$
$$AI = XB^{-1}$$

CRYPTOGRAPHY....

$$A = \begin{bmatrix} 53 & 64 \\ 42 & 56 \\ 90 & 95 \\ 69 & 92 \\ 84 & 89 \\ 111 & 118 \end{bmatrix} \begin{bmatrix} 0.8 & -0.2 \\ -0.6 & 0.4 \end{bmatrix} = \begin{bmatrix} 4 & 15 \\ 0 & 14 \\ 15 & 20 \\ 0 & 23 \\ 15 & 18 \\ 18 & 25 \end{bmatrix}$$

There for matrix A is obtained back and message can be rewritten as 4 15 0 14 15 20 0 23 15 18 18 23.

"DO NOT WORRY"

18. TEACHING LEARNING METHODS

The commonly practiced methods are:

1. Interactive Learning
2. Collaborative Learning
(a) Stump your partner
(b) Fishbowl debate
3. Flipped Classroom
4. Case studies and Problem based Learning
5. ICT
Some other aids like
1. Seminar by students for specific topic
2. Creating Research groups and Clubs
NPTEL Lectures and other Video

19. COURSE CLOSURE REPORT

20. COURSE MATERIAL

MATRICES AND CALCULUS

I YEAR B.Tech, I SEMESTER

	L	T/P/D	C
	3	1/-/-	4

(COMMON TO ALL BRANCHES)

Course Objectives:

1. Determine the rank of the matrix and investigate the solution of system of equations by applying the concepts of consistency.
2. Concepts of Eigen values and Eigen vectors and the nature of quadratic form by finding Eigen values.
3. Concepts of sequence and series and identifying their nature by applying some tests.
4. Mean value theorems geometrical interpretation and their application to the mathematical problems, Evaluation of improper integrals using Beta and Gamma functions
5. Partial differentiation, Total derivative and finding maxima minima of functions of several variables.

Course Outcomes: After learning the contents of this course the students must able to:

1. Write the matrix representation of system of linear equations and identify the consistency of the system of equations.
2. Find the Eigen values and Eigen vectors of the matrix and discuss the nature of the quadratic form.
3. Analyse the convergence of sequence and series.
4. Discuss the applications of mean value theorems to the mathematical problems, Evaluation of improper integrals using Beta and Gamma functions.
5. Examine the extrima of functions of two variables with/ without constraints.

UNIT-I: Matrices and Linear System of Equations

Matrices and Linear system of equations: Real matrices – Symmetric, skew - symmetric, Orthogonal. Complex matrices: Hermitian, Skew – Hermitian and Unitary. Rank-Echelon form, Normal form. Solution of Linear Systems – Gauss Elimination, Gauss Jordan & LU Decomposition methods.

UNIT-II: Eigen Values and Eigen Vectors

Eigen values, Eigen vectors – properties, Cayley-Hamilton Theorem (without Proof) - Inverse and powers of a matrix by Cayley-Hamilton theorem – Diagonolization of matrix- Quadratic forms: Reduction to Canonical form, Nature, Index, Signature.

UNIT-III: Sequences & Series

Basic definitions of Sequences and series, Convergence and divergence, Ratio test, Comparison test, Cauchy's root test, Raabe's test, Integral test ,Absolute and conditional convergence.

UNIT-IV: Beta & Gamma Functions and Mean Value Theorems

Gamma and Beta Functions-Relation between them, their properties – evaluation of improper integrals using Gamma / Beta functions.

Rolle's Theorem, Lagrange's mean value theorem, Cauchy's mean value theorem, Generalized Mean Value theorem (all theorems without proof) – Geometrical interpretation of Mean value theorems.

UNIT-V: Functions of several variables

Partial Differentiation and total differentiation, Functional dependence, Jacobian Determinant- Maxima and Minima of functions of two variables with constraints and without constraints, Method of Lagrange Multipliers.

TEXTBOOKS:

1. B.S. Grewal, Higher Engineering Mathematics, Khanna Publishers, 36th Edition, 2010
2. Advanced Engineering Mathematics by Jain &Iyengar Narosa Pulications
3. Ramana B.V., Higher Engineering Mathematics, Tata McGraw Hill New Delhi, 11th Reprint, 2010.

REFERENCES:

1. G.B. Thomas and R.L. Finney, Calculus and Analytic geometry, 9th Edition, Pearson, Reprint, 2002.
2. Erwin Kreyszig, Advanced Engineering Mathematics, 9th Edition, John Wiley & Sons, 2006.
3. Srimanta Pal and Subodh C. Bhunia, Engineering Mathematics, Oxford University Press, 2015.
4. Advanced Engineering Mathematics (2nd Edition) Michael D.Greenberg

MATRICES AND CALCULUS

I YEAR B.Tech, I SEMESTER	L	T/P/D	C
	3	1/-/-	4

(COMMON TO ALL BRANCHES)

Course Objectives:

1. Determine the rank of the matrix and investigate the solution of system of equations by applying the concepts of consistency.
2. Concepts of Eigen values and Eigen vectors and the nature of quadratic form by finding Eigen values.
3. Concepts of sequence and series and identifying their nature by applying some tests.
4. Mean value theorems geometrical interpretation and their application to the mathematical problems, Evaluation of improper integrals using Beta and Gamma functions
5. Partial differentiation, Total derivative and finding maxima minima of functions of several variables.

Course Outcomes: After learning the contents of this course the students must able to:

1. Write the matrix representation of system of linear equations and identify the consistency of the system of equations.
2. Find the Eigen values and Eigen vectors of the matrix and discuss the nature of the quadratic form.
3. Analyse the convergence of sequence and series.
4. Discuss the applications of mean value theorems to the mathematical problems, Evaluation of improper integrals using Beta and Gamma functions.
5. Examine the extrima of functions of two variables with/ without constraints.

UNIT-I: Matrices and Linear System of Equations

Matrices and Linear system of equations: Real matrices – Symmetric, skew - symmetric, Orthogonal. Complex matrices: Hermitian, Skew – Hermitian and Unitary. Rank-Echelon form, Normal form. Solution of Linear Systems – Gauss Elimination, Gauss Jordan & LU Decomposition methods.

UNIT-II: Eigen Values and Eigen Vectors

Eigen values, Eigen vectors – properties, Cayley-Hamilton Theorem (without Proof) - Inverse and powers of a matrix by Cayley-Hamilton theorem – Diagonolization of matrix- Quadratic forms: Reduction to Canonical form, Nature, Index, Signature.

UNIT-III: Sequences & Series

Basic definitions of Sequences and series, Convergence and divergence, Ratio test, Comparison test, Cauchy's root test, Raabe's test, Integral test ,Absolute and conditional convergence.

UNIT-IV: Beta & Gamma Functions and Mean Value Theorems

Gamma and Beta Functions-Relation between them, their properties – evaluation of improper integrals using Gamma / Beta functions.

Rolle's Theorem, Lagrange's mean value theorem, Cauchy's mean value theorem, Generalized Mean Value theorem (all theorems without proof) – Geometrical interpretation of Mean value theorems.

UNIT-V: Functions of several variables

Partial Differentiation and total differentiation, Functional dependence, Jacobian Determinant- Maxima and Minima of functions of two variables with constraints and without constraints, Method of Lagrange Multipliers.

TEXTBOOKS:

1. B.S. Grewal, Higher Engineering Mathematics, Khanna Publishers, 36th Edition, 2010
2. Advanced Engineering Mathematics by Jain &Iyengar Narosa Pulications
3. Ramana B.V., Higher Engineering Mathematics, Tata McGraw Hill New Delhi, 11th Reprint, 2010.

REFERENCES:

1. G.B. Thomas and R.L. Finney, Calculus and Analytic geometry, 9th Edition, Pearson, Reprint, 2002.
2. Erwin Kreyszig, Advanced Engineering Mathematics, 9th Edition, John Wiley & Sons, 2006.
3. Srimanta Pal and Subodh C. Bhunia, Engineering Mathematics, Oxford University Press, 2015.
4. Advanced Engineering Mathematics (2nd Edition) Michael D.Greenberg

Example: $A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}_{2\times2}, \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}_{3\times3}$

Scalar Matrix:

A matrix 'A' is said to be scalar matrix, if all the diagonal elements are equal.

Example: $A=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}_{2\times2}, \begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 5\end{bmatrix}_{3\times3}$

Upper Triangular matrix:

A square matrix, in which all the elements below the diagonal are zero, is called an upper triangular matrix.

Example; $A=\begin{bmatrix}3 & 5\\ 0 & 3\end{bmatrix}_{2\times2}, \begin{bmatrix}1 & 9 & 8\\ 0 & 2 & 4\\ 0 & 0 & 5\end{bmatrix}_{3\times3}$

Lower Triangular Matrix:

A square matrix, in which all the elements above the diagonal are zero, is called an lower triangular matrix.

Example; $A=\begin{bmatrix}3 & 0\\ 5 & 3\end{bmatrix}_{2\times2}, \begin{bmatrix}1 & 0 & 0\\ 9 & 2 & 0\\ 8 & 7 & 5\end{bmatrix}_{3\times3}$

Triangular Matrix:

A matrix 'A' is said to be Triangular Matrix, if it is either Upper Triangular matrix or Lower Triangular Matrix.

Trace of the Matrix:

The sum of the all the diagonal elements of a square matrix is called the trace of a matrix.

Example: $A=\begin{bmatrix}1 & 5 & 3\\ 9 & 2 & 4\\ 8 & 7 & 5\end{bmatrix}_{3\times3}$ Then Trace of A= 1+2+5=8.

Idempotent Matrix:

A square matrix 'A' is said to be Idempotent, if A^2=A

Example: $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}_{3\times3}$

$$\Rightarrow A\cdot A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}_{3\times3} \cdot \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}_{3\times3} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}_{3\times3}$$

$$\Rightarrow A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}_{3\times3} = A$$

Nilpotent:

If A is a square matrix such that $A^m=0$ is called Nilpotent. If m is the least positive integer such that $A^m=0$ then A is called nilpotent of index.

Example: $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}_{3\times3} \Rightarrow A^3 = 0$

Involutory:

If A is a square matrix such that A^2=I is called Involutory.

Example: $A = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}_{3\times3} \Rightarrow A^2 = I$

Transpose of the matrix:

A matrix obtained by interchanging the rows and columns of a given matrix 'A' is called a Transpose of a Matrix A and is denoted by A^T or A'.

Example: $A = \begin{bmatrix} 1 & 5 & 3 \\ 9 & 2 & 4 \\ 8 & 7 & 5 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 9 & 8 \\ 5 & 2 & 7 \\ 3 & 4 & 5 \end{bmatrix}$

That is if $A = [a_{ij}]_{m\times n}$, then $A^T = [a_{ji}]_{n\times m}$

Note: $(AB)^T = B^T A^T$. Matrix addition and multiplication is associative but not commutative.

Equality:

Two matrices A and B are said to be equal, if they are of the same order and $a_{ij} = b_{ij}$, for every i, j.

Matrix Multiplication:

$C_{m\times n} = A_{m\times p} \cdot B_{p\times n}$ is called the product of the matrices A and B in that order we write C=AB. Where $A = [a_{ik}]_{m\times p}$ & $B = [a_{kj}]_{p\times n}$

That is, if the number of columns in the first matrix equal to the number of rows in the second matrix, then only matrix multiplication is possible.

Inverse of a Matrix:

Inverse of a n-square matrix 'A' is denoted by A^{-1} and is defined such that $AA^{-1} = I = A^{-1}A$ .Where, I is called the Unit matrix.

Result:

- Inverse of a matrix 'A' exists if $|A| \neq 0$, That is if 'A' is non singular matrix only
- Inverse of a matrix 'A' is Unique.
- For a diagonal matrix 'D' with d_{ij} as diagonal elements D^{-1} is a diagonal matrix with reciprocals $\frac{1}{d_{ij}}$ as the diagonal elements.
- Transpose and inverse are commutative. That is $(A^{-1})^T = (A^T)^{-1}$
- $(A^{-1})^{-1} = A$

Adjoint Matrix:

Adjoint matrix of 'A' is obtained by the transpose of a n-square matrix $[A_{ij}]$ where the elements A_{ij} are the cofactors of a_{ij} of A.

Inverse of a matrix 'A': $A^{-1} = \frac{\text{adj} A}{|A|}$ if $|A| \neq 0$.

Real Matrix:

A matrix $A=(a_{ij})$ is said to be real matrix if every element a_{ij} of A is real.

Symmetric Matrix:

A real square matrix is said to be Symmetric, if $A=A^T$. That is $a_{ij}=a_{ji}$.

Skew-Symmetric Matrix:

A real square matrix is said to be Symmetric, if $A^T=-A$. That is $a_{ji}=-a_{ij}$.

Orthogonal matrix:

A real square matrix is said to be Orthogonal, if $A^T=A^{-1}$. That is $AA^T=A^TA=I$

Properties:

- For a skew symmetric matrix $a_{ii}=0$.

  Proof: Since 'A' is a skew symmetric matrix

  That is $A^T=-A \Rightarrow a_{ji}=-a_{ij} \Rightarrow a_{ii}=-a_{ii}$ for $i=j \Rightarrow 2a_{ii}=0 \Rightarrow a_{ii}=0$
- If A matrix 'A' is Symmetric then A^T is Symmetric
- If A matrix 'A' is Symmetric then A^{-1} is Symmetric
- If A and B are symmetric Matrices then A+B and A-B are also Symmetric

  Proof: Since A and B are Symmetric

  That is $A=A^T$ & $B=B^T$

  Now $(A+B)^T=A^T+B^T=A+B$

  And $(A-B)^T=A^T-B^T=A-B$
- If A and B are Symmetric then their product AB is Symmetric ⇔ AB=BA.

  Proof: Given A and B are Symmetric

  That is $A=A^T$ & $B=B^T$

  Suppose AB is Symmetric ⇔ $(AB)^T=AB$

  ⇔ $B^TA^T=AB$

  ⇔ $BA=AB$

  Hence Proved.

- If A and B are symmetric matrices of same order then prove that (i) AB+BA is symmetric and (ii) AB-BA is skew symmetric.

  Proof: Given that 'A' and 'B' are two symmetric matrices.

  So we have $A^T=A$ & $B^T=B$ →(1)

(i) $(AB+BA)^T=(AB)^T+(BA)^T \quad [\because (A+B)^T=A^T+B^T]$

$$=B^TA^T+A^TB^T \quad [\because (AB)^T=B^TA^T]$$

$$=BA+AB \quad [From (1)]$$

$$=AB+BA$$

Therefore $AB+BA$ is symmetric matrix

(ii) $(AB-BA)^T=(AB)^T-(BA)^T \quad [\because (A-B)^T=A^T-B^T]$

$$=B^TA^T-A^TB^T \quad [\because (AB)^T=B^TA^T]$$

$$=BA-AB \quad [From (1)]$$

$$=-(AB-BA)$$

Therefore $AB-BA$ is skew-symmetric matrix

- Every Square matrix can be written as the sum of symmetric matrix and skew symmetric matrix.

Proof: Let A be a square Matrix

Consider $B=\frac{1}{2}(A+A^T)$

Now $B^T=\frac{1}{2}(A+A^T)^T=\frac{1}{2}\left[A^T+(A^T)^T\right]=\frac{1}{2}\left[A^T+A\right]=B$

That is B is a Symmetric matrix

Similarly consider $C=\frac{1}{2}(A-A^T)$

Now $C^T=\frac{1}{2}(A-A^T)^T=\frac{1}{2}\left[A^T-(A^T)^T\right]=-\frac{1}{2}\left[A^T-A\right]=-C$

That is 'C' is a skew-symmetric matrix

Now $B+C=\frac{1}{2}(A+A^T)+\frac{1}{2}(A-A^T)$

$$=\frac{1}{2}\left[A+A^T+A-A^T\right]=\frac{2A}{2}=A$$

Hence every square matrix can be expressed as sum of symmetric and skew symmetric matrices

- If A and B are two orthogonal matrices with order n, then AB and BA are orthogonal matrices.

Proof: Since A and B are both orthogonal matrices

Therefore $AA^T=A^TA=I \quad \rightarrow(1)$

$BB^T=B^TB=I \quad \rightarrow(2)$

We have $(AB)^T=B^TA^T \quad \rightarrow(3)$

Now $(AB)^T(AB)=(B^TA^T)(AB)$

$= B^T(A^TA)B$ [*Using Associate Property*]

$= B^T(I)B$ [From equation (1)]

$= B^TB = I$ [From equation (2)]

Therefore AB is an orthogonal matrix

Similarly we can prove that BA is also an orthogonal matrix.

- The determinant of orthogonal matrix is ± 1.

  Solution: Let 'A' be an orthogonal matrix

  That is $A^T = A^{-1}$

  $$\Rightarrow A^TA = I \Rightarrow |A^TA| = |I| \Rightarrow |A^T||A| = 1$$

  $$\Rightarrow |A||A| = 1 \Rightarrow |A|^2 = 1$$

  $$\Rightarrow |A| = \pm 1$$

- Prove that inverse of an orthogonal matrix is orthogonal and its transpose is also an orthogonal matrix.

  Proof: Given A is an orthogonal matrix.

  That is $A^T = A^{-1}$

  Consider $A^T \cdot A = I$

  Taking inverse on both sides,

  $(A^T \cdot A)^{-1} = I^{-1}$

  $\Rightarrow A^{-1}(A^T)^{-1} = I$ $\left[\text{Since } (AB)^{-1} = B^{-1}A^{-1}\right]$

  $\Rightarrow A^{-1}(A^{-1})^T = I$ $\left[\text{Since } (A^T)^{-1} = (A^{-1})^T\right]$.

  Therefore A^{-1} is an orthogonal matrix.

  Again $A^T \cdot A = I$

  Taking Transpose on both sides,

  $(A^T \cdot A)^T = I^T$

  $\Rightarrow A^T(A^T)^T = I$ $\left[\text{Since } (AB)^T = B^TA^T\right]$

  $\Rightarrow A^TA = I.$ $\left[\text{Since } (A^T)^T = A\right]$

  Therefore A^{-1} is an orthogonal matrix.

- Product of two orthogonal matrices is Orthogonal.

  Proof: Let 'A' and 'B' be two orthogonal matrices.

  That is $A^T = A^{-1}$ & $B^T = B^{-1}$

  Now $(AB)(AB)^T = (AB)(B^TA^T)$

$$\Rightarrow (AB)(AB)^T = A(BB^T)A^T$$
$$\Rightarrow (AB)(AB)^T = A(I)A^T = AA^T = I$$
$$\Rightarrow (AB)(AB)^T = I$$
$$\Rightarrow (AB)^T = (AB)^{-1}$$

So, product of two orthogonal matrices is orthogonal.

Complex Matrix:

A matrix whose elements are complex are real numbers is called a complex matrix.

Conjugate of a Matrix:

A matrix obtained from another matrix 'A' by replacing the elements of 'A' with their complex conjugate is called conjugate matrix of a complex matrix 'A'.

Example: $A = \begin{bmatrix} 2+3i & 5 \\ 6-7i & 5+i \end{bmatrix} \Rightarrow \overline{A} = \begin{bmatrix} 2-3i & 5 \\ 6+7i & 5-i \end{bmatrix}$

Conjugate Transpose of a Matrix:

The transpose of a conjugate matrix of a complex matrix 'A' is called the transpose of that matrix.

In general the conjugate transpose of a matrix denoted by $\overline{A}^T$ or A^θ.

Example: $A = \begin{bmatrix} 2+3i & 5 \\ 6-7i & 5+i \end{bmatrix} \Rightarrow \overline{A} = \begin{bmatrix} 2-3i & 5 \\ 6+7i & 5-i \end{bmatrix} \Rightarrow \overline{A}^T = A^\theta = \begin{bmatrix} 2-3i & 6+7i \\ 5 & 5-i \end{bmatrix}$

Hermitian Matrix:

A square matrix A is said to be a Hermitian Matrix if $A^\theta = A$ or $\overline{A}^T = A$.

Example:

$$A = \begin{bmatrix} 1 & 2-i & 3+i \\ 2+i & 2 & -7i \\ 3-i & 7i & 5 \end{bmatrix}, \qquad \overline{A} = \begin{bmatrix} 1 & 2+i & 3-i \\ 2-i & 2 & 7i \\ 3+i & -7i & 5 \end{bmatrix}$$

$$\Rightarrow \overline{A}^T = \begin{bmatrix} 1 & 2-i & 3+i \\ 2+i & 2 & -7i \\ 3-i & 7i & 5 \end{bmatrix} = A$$

A is Hermitian Matrix

Skew-Hermitian Matrix:

A square matrix A is said to be a Skew-Hermitian Matrix if $A^{\theta} = -A$ or $\overline{A}^{T} = -A$.

Example: $A = \begin{bmatrix} 2i & 3-i & 4+3i \\ -3-i & -4i & 6+5i \\ 3i-4 & 5i-6 & 0 \end{bmatrix}$

Unitary Matrix:

A square matrix A is said to be a Unitary Matrix if $A^{\theta} = A^{-1}$ or $\overline{A}^{T} = A^{-1}$ or $A\overline{A}^{T} = \overline{A}^{T}A = I$.

Example: $A = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \Rightarrow \overline{A}^{T} = \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} \Rightarrow A\overline{A}^{T} = AA^{\theta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Problems:

1. Express the matrix A as sum of symmetric and skew symmetric matrices where $A = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$

Matrices And Linear System of equations

Matrix: A matrix is a rectangular array of m·n numbers arranged in m rows and n-columns [Vertical lines]. These numbers known as elements or entries are enclosed in brackets [ ] or ( ) or || ||

The order of such matrix is m×n and is said to be a rectangular matrix.

Note:- Elements of the matrix are located by the double ij where i denotes the row and j denotes the column

Equality:- Two matrices A and B are ~~enclosed in brackets~~ equal if they are of the same order and $a_{ij} = b_{ij}$, for every i, j

Matrix Multiplication:- $C_{m\times n} = A_{m\times p} B_{p\times n}$ where $c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$

Transpose of a Matrix: Transpose of a matrix $A^T_{m\times n}$ is denoted by interchanging rows and columns.

Result:- $(AB)^T = B^T A^T$

Matrix addition and multiplication is associative but not commutative.

Inverse of a matrix :-

Consider only square matrix

Inverse of a n-square matrix A is denoted by A^{-1} and is defined such that $AA^{-1} = A^{-1}A = I$ where I is n×n unit matrix.

300 7/12 (600)

Result: Inverse of A exists only if [illegible] i.e A is non singular

Result: Inverse of A a matrix is unique

Result:- Inverse of a product is the product of inverse in the reverse order. i.e $(AB)^{-1} = B^{-1}A^{-1}$

Result:- For a diagonal matrix D with d_{ij} as diagonal elements D^{-1} is a diagonal matrix with reciprocals $1/d_{ij}$ as the diagonal element.

Result:- Transposition and inverse are commutative.

i.e $(A^{-1})^T = (A^T)^{-1}$

Result:- $(A^{-1})^{-1} = A$

Adjoint Matrix :-

A is denoted by adj A is the transpose of a n- square matrix $[A_{ij}]$ where the elements A_{ij} are the cofactors of a_{ij} of A

i.e if ~~cofac~~ adj A = if Cofactors of A_{ij}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & & a_{nn} \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}^T$$

$$= \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

Result: Adj (AB) = (Adj A) (Adj B)

Inverse of a matrix from the adj

$$\text{i.e } A^{-1} = \frac{\text{adj } A}{\det A}$$

1. Find the inverse of a matrix by adj matrix

1. $$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

Sol:- $\det A = |A| = 1(-12-12) - 1(-4+12) - 2(-3-9)$

$= -24 - 8 + 24 = -8 \neq 0$

$|A| \neq 0$, inverse of A exists

adj A = [cofactors of a_{ij} of A]^T

i.e cofactor matrix of A $= \begin{bmatrix} (-12-12) & -(-4-6) & (-4+6) \\ -(-4+12) & (-4+6) & -(-4+2) \\ (-3-9) & -(-3-3) & (3-1) \end{bmatrix} = \begin{bmatrix} -24 & 10 & 2 \\ -8 & 2 & 2 \\ -12 & 6 & 2 \end{bmatrix}$

∴ Adj of A = Transpose of a cofactor matrix of A

$$= \begin{bmatrix} -24 & 10 & 2 \\ -8 & 2 & 2 \\ -12 & 6 & 2 \end{bmatrix}^T = \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\text{Adj } A}{|A|}$$

$$= \frac{-1}{8}\begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & 1 \end{bmatrix}$$

Def:-

Let A be a rectangular matrix of order $m \times n$. Submatrix of a matrix A is any matrix obtained from A by omitting some rows and columns in A.

A is a submatrix of itself.

Rank :-

Rank of a matrix A is the positive integer 'r' such that there exists atleast one r-rowed square matrix with non-vanishing determinent while every $(r+1)$ or more rowed matrices have vanishing determinents.

Thus the rank of a matrix is the largest order of a non-zero minor of matrix.

Rank of A is denoted by $r(A)$

Result :- Rank of A and A^T is same

Note 1:- Rank of null matrix is zero

Note 2:- For a rectangular matrix A of order $m \times n$ rank of $A \leq \min(m, n)$ i.e rank cannot exceed the smaller then m and n

Note 3:- For a n square matrix, if rank = n then $|A| \neq 0$,

i.e, A is non-singular.

Note 4:- For any square matrix, if rank < n, then $|A| = 0$. i.e

A is singular.

Elementary Row Transpose on a Matrix

1. R_{ij}: Interchange of the i^{th} and j^{th} rows and ~~columns~~
2. $R_{i(k)}$: Multiplication of every element of i^{th} row by a non-zero scalar k.
3. $R_{ij(k)}$: Addition to the elements of i^{th} row, of k times the corresponding elements of the j^{th} row

In a similar way, elementary column transformations/operatn are denoted by C_{ij}, $C_{i(k)}$, $C_{ij(k)}$ where the row in the above definitions is replaced by column

Example :-

Inverse by Gauss-Jordon Method:

1. Find the inverse of A by Gauss-Jordon method where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

Sol:- ~~Find~~ Consider A/I and apply elementary row operations on both A and I until A gets transformed to I

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right] \underset{\substack{R_{21(-2)} \\ R_{31(-3)}}}{\sim} \left[\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right]$$

$$\underset{\substack{R_{23} \\ R_{2(-1)} \\ R_{3(-1)}}}{\sim} \left[\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$\begin{array}{l} R_{23}(-3) \\ R_{13}(-3) \end{array} \sim \left[\begin{array}{cccccc} 1 & 2 & 0 & -5 & 3 & 0 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$R_{12}(-2) \sim \left[\begin{array}{cccccc} 1 & 0 & 0 & -1 & -3 & -2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right] = I/A^{-1}$$

$$\text{Thus } A^{-1} = \begin{bmatrix} -1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

— x —

2. $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Sol:- Consider A/I and apply elementary row operations on both A and I until A gets transformed to I

$$\left[\begin{array}{cccccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \underset{\begin{array}{l} R_{21}(-1) \\ R_{31}(-1) \end{array}}{\sim} \left[\begin{array}{cccccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$R_{13}(-3) \sim \left[\begin{array}{cccccc} 1 & 3 & 0 & 4 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$R_{12}(-3) \sim \left[\begin{array}{cccccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\text{Thus } A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\begin{vmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{vmatrix}$$

Sol: Consider A/I and apply elementary row or column operations on A & I until A goes to transformed to I

$$A/I = \begin{bmatrix} -1 & -3 & 3 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 2 & -5 & 2 & -3 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} R_{21}(1) \\ R_{31}(+2) \\ R_{41}(-1) \end{matrix} \sim \begin{bmatrix} -1 & -3 & 3 & -1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 2 & -1 & 1 & 1 & 0 & 0 \\ 0 & -11 & 8 & -5 & 2 & 0 & 1 & 0 \\ 0 & 4 & -3 & 2 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{23} \begin{bmatrix} -1 & -3 & 3 & -1 & 1 & 0 & 0 & 0 \\ 0 & -11 & 8 & -5 & 2 & 0 & 1 & 0 \\ 0 & -2 & 2 & -1 & 1 & 1 & 0 & 0 \\ 0 & 4 & -3 & 2 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{23}(-6) \begin{bmatrix} -1 & -3 & 3 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 1 & -4 & -6 & 1 & 0 \\ 0 & -2 & 2 & -1 & 1 & 1 & 0 & 0 \\ 0 & 4 & -3 & 2 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{32(2)},\ R_{42(-4)} \quad \begin{bmatrix} -1 & -3 & 3 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 1 & -4 & -6 & 1 & 0 \\ 0 & 0 & -6 & +1 & -7 & -11 & 2 & 0 \\ 0 & 0 & 13 & -2 & 15 & 24 & -4 & 1 \end{bmatrix}$$

$$R_{43} \sim \begin{bmatrix} -1 & -3 & 3 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 1 & -4 & -6 & 1 & 0 \\ 0 & 0 & 13 & -2 & 15 & 24 & -4 & 1 \\ 0 & 0 & -6 & +1 & -7 & -11 & 2 & 0 \end{bmatrix}$$

$$R_{34(2)} \quad \begin{bmatrix} -1 & -3 & 3 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 1 & -4 & -6 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & -6 & +1 & -7 & -11 & 2 & 0 \end{bmatrix}$$

$$R_{43(6)} \quad \begin{bmatrix} -1 & -3 & 3 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 1 & -4 & -6 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & +1 & -1 & 1 & 2 & 6 \end{bmatrix}$$

$$R_{14(1)},\ R_{24(-1)} \quad \begin{bmatrix} -1 & -3 & 3 & 0 & 0 & 1 & 2 & 6 \\ 0 & 1 & -4 & 0 & -3 & -7 & -1 & -6 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 2 & 6 \end{bmatrix}$$

$R_{13(-3)}$
$R_{23(4)}$
$$\begin{bmatrix} -1 & -3 & 0 & 0 & -3 & -5 & 2 & 3 \\ 0 & 1 & 0 & 0 & 1 & 1 & -1 & -2 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 2 & 6 \end{bmatrix}$$

$R_{12(3)}$
$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & -2 & -1 & -3 \\ 0 & 1 & 0 & 0 & 1 & 1 & -1 & -2 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 2 & 6 \end{bmatrix}$$

$R_{1(-1)}$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 1 & 3 \\ 0 & 1 & 0 & 0 & 1 & 1 & -1 & -2 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 2 & 6 \end{bmatrix} = [I / A^{-1}]$$

$$\therefore A^{-1} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$$ Ans.

Kaushik

Find the inverse of the matrix A by adjoint [illegible]

1. $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$ (3) $\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$ (4) $\begin{bmatrix} 7 & 6 & 2 \\ -1 & 2 & 4 \\ 3 & 3 & 8 \end{bmatrix}$

By Gauss-Jordan Method:-

1. $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ (2) $\begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Equivalent Matrices:

Two matrices A and B are said to be equivalent, denoted by $A \sim B$, if one matrix say A can be obtained from B by sequence of elementery transformation

Row - equivalent:-

Two matrices A and B are said to be row-equivalent if A can be reduced to B by a sequence of elementary row transformations or viceversa.

Determination of Rank of a Matrix A:-

Let A be a rectangular matrix of order $m \times n$

1. Enumeration:- Evaluate all the minors such that a minor of r is non-zero and every minor of $(r+1)$ or more is zero.

Note: This is impractible for matrices of higher order

Note:- Minor of a Matrix:

let A be a $m \times n$ matrix. The determinant of a square submatrix of A is called a minor of the matrix. If the order of the square sub-matrix is t then its determinent is called a minor of order of t.

2. Apply only elementary row operations on A. Then the number of non-zero rows is the Rank of A.

III. Normal form N:- Normal form N of a matrix A of rank r is one of the forms

$$N = I_r, \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, [I_r, 0], \begin{bmatrix} I_r \\ 0 \end{bmatrix}$$

Where I_r is an identity matrix of order r By the application of both elementery row and column operations, a matrix of rank r can be reduced to normal forms Then the rank of A is r.

IV:- Echelon form:-

Row reduced Echelon form: The number of non-zero rows in an Echelon form is the rank.

Result:- Equivalent matrices have the same order and same rank because elementery transformations do not after (effect) its order and rank.

Determine the rank of the following matrices

Example - $A = \begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -1.5 \end{bmatrix}$

Sol:- Apply elementary row operations on A

$$A = \begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -1.5 \end{bmatrix} \underset{R_{31}(\frac{1}{2})}{\overset{R_{21}(-2)}{\sim}} \begin{bmatrix} 4 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

∴ The no of non-zero rows are only one. So the rank of A is one.

Exe ②

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\underset{R_{31}(-3)}{\overset{R_{21}(-2)}{\sim}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & -5 & 7 \end{bmatrix} \underset{R_{32}(-5)}{\sim} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 18 \end{bmatrix}$$

∴ The no of non zero rows of A are 3.

So the rank of A is 3.

3) $$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix} \underset{R_{31}(-3)}{\overset{R_{21}(-1)}{\sim}} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & -2 & 3 & -2 \end{bmatrix} \overset{R_{32}(-1)}{\sim} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ The no of non zero rows of A is 2. So the Rank of A is 2

Assignment problems

(i) $\begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -1.5 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ 2 & 13 & 10 \end{bmatrix}$

( ) Find the rank of A & rank of rank of A+B, AB, BA

$B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$

(4) $\begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{bmatrix}$

(5) $\begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ 6 & 2 & 2 & 2 \\ 9 & 9 & 6 & 3 \end{bmatrix}$

(6) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

(7) $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$

(8) $\begin{bmatrix} 3 & -2 & 0 & -1 & -7 \\ 0 & 2 & 2 & 1 & -5 \\ 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 & 6 \end{bmatrix}$

(9) $\begin{bmatrix} 3 & -2 & 0 & -1 & -7 \\ 0 & 2 & 2 & 1 & -5 \\ 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 & 6 \end{bmatrix}$

(10) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

(11) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}$

(12) $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$

(13) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

(14) $\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$

(15) $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}$

(16) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ (17) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -2 & -3 & -1 \end{bmatrix}$ (18) $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$

(19) $\begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 3 & 11 & 6 \end{bmatrix}$ (20) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & 7 \end{bmatrix}$

Reduce A to Echelon form and then to its row canonical form [or reduced Echelon form] where

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix}$$

Sol:- Applying elementary row operations on A

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix} \underset{R_{41(-4)}}{\overset{R_{31(-2)}}{\sim}} \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & -11 & 5 & -3 \\ 0 & -11 & 5 & -3 \end{bmatrix}$$

$$\underset{R_{42(-1)}}{\overset{R_{32(-1)}}{\sim}} \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_{2/11} \sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -5/11 & 3/11 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_{12(-3)} \sim \begin{bmatrix} 1 & 0 & \frac{4}{11} & \frac{13}{11} \\ 0 & 1 & -\frac{5}{11} & \frac{3}{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ This is the row canonical or row reduced Echelon form

Rank of A is 2. Because no of non zero rows of A are 2

2. $\begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix}$

Sol:- $A = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix} \underset{R_{31}(-3)}{\overset{R_{21}(-2)}{\sim}} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 7 & -7 & 6 \end{bmatrix}$

$\underset{R_{12}(2/3)}{\overset{R_{32}(-7/3)}{\sim}} \begin{bmatrix} 1 & 0 & 1/3 & 5/3 \\ 0 & 3 & -4 & 4 \\ 0 & 0 & 7/3 & -\frac{10}{3} \end{bmatrix} \underset{R_{2}/3}{\sim}$

$R_2(1/3) \begin{bmatrix} 1 & 0 & 1/3 & 5/3 \\ 0 & 1 & -4/3 & 4/3 \\ 0 & 0 & 7/3 & -10/3 \end{bmatrix} R_3(3/7) \begin{bmatrix} 1 & 0 & 1/3 & 5/3 \\ 0 & 1 & -4/3 & 4/3 \\ 0 & 0 & 1 & -10/7 \end{bmatrix}$

$\underset{R_{23}(4/3)}{R_{13}(-1/3)} \begin{bmatrix} 1 & 0 & 0 & \frac{5}{7} \\ 0 & 1 & 0 & -4/7 \\ 0 & 0 & 1 & -10/7 \end{bmatrix}$

∴ The no of non-zero rows are 3

$\therefore r(A) = 3$

Echelon form:-

$A = \begin{bmatrix} 1 & 2 & -5 \\ -4 & 1 & -6 \\ 6 & 3 & -4 \end{bmatrix}$

Sol:- Apply row elementary operations

$$A = \begin{bmatrix} 1 & 2 & -5 \\ -4 & 1 & -6 \\ 6 & 3 & -4 \end{bmatrix} \begin{matrix} R_{21}(4) \\ R_{31}(-6) \end{matrix} \sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & -9 & 26 \end{bmatrix} \underset{R_{32}(-1)}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_{2}(1/9) \sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & -26/9 \\ 0 & 0 & 0 \end{bmatrix} R_{12}(-2) \sim \begin{bmatrix} 1 & 0 & -7/9 \\ 0 & 1 & -26/9 \\ 0 & 0 & 0 \end{bmatrix}$$

∴ The no of non-zero rows of A are 2

∴ $r(A) = 2$

4. $\begin{bmatrix} 2 & 3 & -2 & 5 & 1 \\ 3 & -1 & 2 & 0 & 4 \\ 4 & -5 & 6 & -5 & 7 \end{bmatrix}$

Sol:- Apply row elementary operations on A

$$A = \begin{bmatrix} 2 & 3 & -2 & 5 & 1 \\ 3 & -1 & 2 & 0 & 4 \\ 4 & -5 & 6 & -5 & 7 \end{bmatrix} \begin{matrix} R_{21}(-3/2) \\ R_{41}(-2) \end{matrix} \sim \begin{bmatrix} 2 & 3 & -2 & 5 & 1 \\ 0 & -11/2 & 5 & -\frac{15}{2} & \frac{5}{2} \\ 0 & -11 & 10 & -15 & +5 \end{bmatrix}$$

$$\begin{matrix} R_{32}(-2) \\ R_{23} \end{matrix} \sim \begin{bmatrix} 2 & 3 & -2 & 5 & 1 \\ 0 & -11 & 10 & -15 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \underset{R_{2}(-11)}{\sim} \begin{bmatrix} 2 & 3 & -2 & 5 & 1 \\ 0 & 1 & -10/11 & \frac{15}{11} & \frac{-5}{11} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_{12}(-3) \sim \begin{bmatrix} 2 & 0 & \frac{8}{11} & \frac{10}{11} & \frac{-4}{11} \\ 0 & 1 & \frac{-10}{11} & \frac{15}{11} & \frac{-5}{11} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \underset{R_{1}(1/2)}{\sim} \begin{bmatrix} 1 & 0 & \frac{4}{11} & \frac{5}{11} & \frac{-2}{11} \\ 0 & 1 & -\frac{10}{11} & \frac{15}{11} & -\frac{5}{11} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ The Rank of A is 2

Assignment problems

① Reduce the matrices A into echelon form and hence find its Rank.

1. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ $r(A) = 3$

② $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$ $r(A) = 3$

③ $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$ $r(A) = 2$

4. $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$ $r(A) = 4$

5. $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ $r(A) = 2$ $B = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$ $r(B) = 1$

find the rank of A, B, $A+B$, AB and BA

$r(A+B) = 2$, $r(AB) = 0$, $r(BA) = 1$

6. Find the rank of the matrix $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$ $r(A)\ 3$

7. Define the rank of the matrix and find the rank of the following matrix $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$ 2005

$r(A) = 2$

8. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ $r(A) = 3$

Normal form:-

Procedure to obtain Normal form

Consider $A_{m\times n} = I_{m\times m} A_{m\times n} I_{n\times n}$

Apply elementary row operations on A and on the prefactor $I_{m\times n}$ and apply elementary column operations on A and post-factor $I_{n\times n}$ such that A on the L.H.S reduces to Normal form. Then $I_{m\times m}$ reduces to $P_{m\times m}$ and $I_{n\times n}$ reduces to $Q_{n\times n}$: resulting in $N = PAQ$

Here P and Q are non-singular matrices

Thus for any matrix of rank r, there exist non-singular matrices P and Q such that

$$PAQ = N = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Example: Find the non-singular matrices P and Q such that the normal form of A is PAQ where

$$A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}_{3\times 4}$$ Hence find the rank.

Sol:- Consider $A_{3\times 4} = I_{3\times 3} A_{3\times 4} I_{4\times 4}$

$$\begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

prefactor $R_{21(-1)}$, $R_{31(-1)}$

$$\begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

prefactor $R_{32(-2)}$

$$\begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

postfactor $C_{21(-3)}$, $C_{31(-6)}$, $C_{41(1)}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -6 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

~~C_{41}~~

postfactor $C_{32(1)}$, $C_{42(-2)}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -9 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ.$

where $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -3 & -9 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

And hence rank of A is 2

Example 2:- $\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ Ans $P = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1/2 & 0 \\ -1 & -2 & 1 \end{bmatrix}$ $Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

rank of A = 2

1.
$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}_{3\times3}$$

Sol: Consider $A_{3\times3} = I_{3\times3} \, A_{3\times3} \, I_{3\times3}$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{21(-1)} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{32(1)} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & +1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} C_{21(-1)} \\ C_{31(-2)} \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & +1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_{32(-1)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & +1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus 1b

Thus the L.H.S is in the normal form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

where $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & H & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

4. $A = \begin{matrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \end{matrix}$ (struck out)

4 $A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 4 & 2 & 2 & -1 \\ 2 & 2 & 0 & -2 \end{bmatrix}_{3\times4}$

Consider $A_{3\times4} = I_{3\times3} \, A_{3\times4} \, I_{4\times4}$

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 4 & 2 & 2 & -1 \\ 2 & 2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_{21}(-4)$
$R_{31}(-2)$
$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 6 & 6 & -9 \\ 0 & 4 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2(1/6)$
$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & 1 & -3/2 \\ 0 & 4 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{4}{6} & \frac{1}{6} & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_{32}(-4)$
$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & 1 & -3/2 \\ 0 & 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1/6 & 0 \\ 2/3 & -2/3 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

OM SAIRAM

$$R_3(-1/2)\begin{bmatrix}1&-1&-1&2\\0&1&1&-3/2\\0&0&1&0\end{bmatrix}=\begin{bmatrix}1&0&0\\-2/3&1/6&0\\-1/3&+1/3&-1/2\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$\begin{matrix}C_{21}(1)\\C_{31}(1)\\C_{41}(-2)\end{matrix}\begin{bmatrix}1&0&0&0\\0&1&1&-3/2\\0&0&1&0\end{bmatrix}=\begin{bmatrix}1&0&0\\-2/3&1/6&0\\-1/3&+1/3&-1/2\end{bmatrix}A\begin{bmatrix}1&1&1&-2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$\begin{matrix}C_{32}(-1)\\C_{42}(3/2)\end{matrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}=\begin{bmatrix}1&0&0\\-2/3&1/6&0\\-1/3&+1/3&-1/2\end{bmatrix}A\begin{bmatrix}1&1&0&-1/2\\0&1&-1&3/2\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

Thus the L.H.S is in the Normal form. $[I_3 : 0]$ ie $N = PAQ$

$[I_3\ 0]$ .. where $P=\begin{bmatrix}1&0&0\\-2/3&1/6&0\\-1/3&+1/3&-1/2\end{bmatrix}$ $Q=\begin{bmatrix}1&1&0&-1/2\\0&1&-1&3/2\\0&0&1&0\\0&0&0&1\end{bmatrix}$

Assignment problems:-

1) $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$

(2) $A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & +2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$

(3) $A = \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$

(4) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 1 & 2 \end{pmatrix}$

(5) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ $r(A)=2$

JNTU 2002

(6) $A = \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$ $r(A)=2$

(7) $A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{bmatrix}$ $r(A)=2$

(8) $A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$ $r(A)=2$

(9) $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ $r(A)=2$

(10) $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 2 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ $r(A) = 3$ JNTU-2002

(11) Find the non-

$A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$ $r(A)=2$

(12) (i) $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$ $r(A)=2$ JNTU 2003 2005

(ii) $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ JNTU 2004.

(13) $A \cdot \begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -4 \\ 3 & 3 & -6 \end{bmatrix}$ $r(A)=2$

(14) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 1 & 0 & 2 \\ 1 & 4 & 5 & 3 \end{bmatrix}$ $r(A) = 2$

System of Linear Alegebraic Equations:-

A system of m - linear equations in n-unknowns $x_1, x_2, \ldots x_n$ is a set of equations of the form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots \qquad \rightarrow (1)$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Non - Homogeneous System:-

When all b_i's are not zero, i.e at least one b_i is non-zero.

Homogeneous System:- If $b_i = 0, i = 1$ to m (all RHS constants are zero)

Solution:- Solution of system (1) is a set of numbers $x_1, x_2 \ldots x_n$ which statistics all the equations of the system (1)

Trivial Solution:-

Trivial solution is a solution where all x_i are zero

i.e $x_1 = x_2 = \cdots = x_n = 0$

let $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times1}$ ← Solution (vector), $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times1}$ → (RHS) Constant Vector — be two column vectors

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$ this is coefficient matrix of the system (1)

The system (1) can be represented as $A_{m\times n} X_{n\times1} = B_{m\times1}$

Augmented Matrix :- [A/B] or $\tilde{A}$ of system ① is obtained by augmenting A by the column B

i.e $\tilde{A} = [A/B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & & \cdots & \cdots \\ \cdots & \cdots & & \cdots & \cdots \\ \cdots & \cdots & & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$

Consistent :- System is said to be consistent if (1) has at least one solution.

If system has no solution at all then that system is inconsist

Solution of Linear System of Linear Equations :-

We consider two methods of obtaining solution of system of m - linear equation in n-unknowns. They are

1. Cramer's rule 2. Matrix inverse

Cramer's rule :- [Solution by determinants]

a. If A is non-singular i.e D = determinant of A = |A| ≠ 0

Then system (1) has a unique solution given by

$x_i = \frac{D_i}{D}$ for $i = 1, 2, \ldots n$

where D_i is the determinant obtained from D by replacing the i^{th} column in D by constant column vector B.

b. For homogeneous system with D ≠ 0 only trivial solution exist

c. For homogeneous system with D = 0, non-trivial solution exist

Note :- Cramer's rule is not suitable for computations

Matrix Inversion Method:-

Consider the system of n-equations in n-unknowns represented by $AX = B$

where A is n-square non-singular matrix. Premultiplying by A^{-1} on the sides. we get

or $A^{-1}AX = A^{-1}B$

$X = A^{-1}B$ which is required Solution.

Here A^{-1}, the inverse of A is obtained by Gauss Jordon method.

Consider A/I

Apply only elementary row operations on both A and I such that A is reduced to an identity matrix I, then I gets transformed to A^{-1}

Consistency of System of Linear equations:-

Fundamental Theorem:-

I. If rank of A and rank of the augmented matrix $\tilde{A}$ are equal, then the system is consistants

(a) If $r(A) = r(\tilde{A}) = n$

Then unique solution exists.

(b) If $r(A) = r(\tilde{A}) < n$

then infinitely many solutions exist in terms of $(n-r)$ arbitorary constants

II. If rank of A is not equal to rank of $\tilde{A}$ then the system is inconsistent and has no solution at all.

procedure:-

1. Determine $r(A)$ and $r(\tilde{A})$

2. If $r(A) \neq r(\tilde{A})$ system inconsistent, no solutions.

3. If $r(A) = r(\tilde{A}) = n$.

Then the solution may be obtained by Cramer rule or matrix inversion method.

4. If $r(A) = r(\tilde{A}) < n$

Then rewrite $x_1, \ldots, x_r$ variables (whose coefficient submatrix has rank r) in terms $(n-r)$ variable and solve by Gaussian elimination of Gauss-Jordan elimination method.

$x_1 + x_2 - 2x_3 + x_4 + 3x_5 = 1,$
$2x_1 - x_2 + 2x_3 + 2x_4 + 6x_5 = 2,$
$3x_1 + 2x_2 - 4x_3 - 3x_4 - 9x_5 = 3.$

Sol:- By applying elementary row operations

$$[A/B] = \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 1 & 3 & 1 \\ 2 & -1 & 2 & 2 & 6 & 2 \\ 3 & 2 & -4 & -3 & -9 & 3 \end{array}\right]$$

~~$R_{21(-2)}$ 1 1 -2 1 3 1~~
~~$R_{31(-3)}$ 2 -1 2 2~~

$$\begin{matrix} R_{21(-2)} \\ R_{31(-3)} \end{matrix} \sim \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 1 & 3 & 1 \\ 0 & -3 & 6 & 0 & 0 & 0 \\ 0 & -1 & +2 & -6 & -18 & 0 \end{array}\right]$$

$$R_{32(-3)} \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 1 & 3 & 1 \\ 0 & -3 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 & -18 & 0 \end{array}\right]_{3\times5}$$

$\therefore$ Rank of $A = 3 = r(A|B) < 5 = n$ = no of variables.

So, the System is Consistant but the infinite number of Solutions. choosing $n-r = 5-3 = 2$

By back Substitution:-

$x_5 = b \Rightarrow -6x_4 - 18x_5 = 0$

$\Rightarrow 6x_4 = -18x_5$

$\Rightarrow x_4 = -3b$

$x_3 = a \Rightarrow -3x_2 + 6x_3 = 0$

$\Rightarrow 3x_2 = 6x_3$

$x_2 = 2a$

$\therefore x_1 = 1, x_2 = 2a, x_3 = a$
$x_4 = 3b, x_5 = b$

$x_1 + x_2 - 2x_3 + x_4 + 3x_5 = 1$

$x_1 + 2a - 2a - 3b + 3b = 1$

$\therefore x_1 = 1$

2. $x_1 + x_2 + 2x_3 + x_4 = 5$

~~$2x_1 + x_2 + 2x_3 + 6$~~

$2x_1 + 3x_2 - x_3 - 2x_4 = 2$

$4x_1 + 5x_2 + 3x_3 = 7$

Sol:- Apply elementary row operations on $[A|B]$

$$[A|B] = \begin{bmatrix} 1 & 1 & 2 & 1 & | & 5 \\ 2 & 3 & -1 & -2 & | & 2 \\ 4 & 5 & 3 & 0 & | & 7 \end{bmatrix}$$

$$\underset{R_{31(-4)}}{R_{21(-2)}} \sim \begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 1 & -5 & -4 & -13 \end{bmatrix} \underset{R_{32(-1)}}{\sim} \begin{bmatrix} 1 & 1 & 2 & 1 & | & 5 \\ 0 & 1 & -5 & -4 & | & -8 \\ 0 & 0 & 0 & 0 & | & -5 \end{bmatrix}$$

rank of $A = 2 \neq 3 =$ rank of $(A|B)$

So, the given system is inconsistent and therefore A has no solutions.

3. $-x_1 + x_2 + 2x_3 = 2.$

$3x_1 - x_2 + x_3 = 6.$

$-x_1 + 3x_2 + 4x_3 = 4.$

Sol:- Apply elementary row operations on $[A|B]$

$$[A|B] = \begin{bmatrix} -1 & 1 & 2 & 2 \\ 3 & -1 & 1 & 6 \\ -1 & 3 & 4 & 4 \end{bmatrix} \underset{R_{31(-1)}}{R_{21(3)}} \sim \begin{bmatrix} -1 & 1 & 2 & 2 \\ 0 & +2 & 7 & 12 \\ 0 & 2 & 2 & 2 \end{bmatrix}$$

$$R_{2(1/2)} \begin{bmatrix} -1 & 1 & 2 & 2 \\ 0 & 1 & +7/2 & +6 \\ 0 & 2 & 2 & 2 \end{bmatrix} R_{32(-2)} \begin{bmatrix} -1 & 1 & 2 & 2 \\ 0 & 1 & +7/2 & +6 \\ 0 & 0 & -5 & -10 \end{bmatrix}$$

$$R_{3(-1/5)} \begin{bmatrix} -1 & 1 & 2 & | & 2 \\ 0 & 1 & 7/2 & | & 6 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}$$

By back substitution

$x_3 = 2$

$x_2 + 7/2\, x_3 = 6$

$\Rightarrow x_2 + 7 = 6$

$\Rightarrow x_2 = -1$

$-x_1 + x_2 + 2x_3 = 2 \Rightarrow -x_1 - 1 + 4 = 2$

$\Rightarrow -x_1 = -1$

$\Rightarrow x_1 = 1$

$r(A) = 3 = r(A|B) = n =$ no of variables

$\therefore x_1 = 1, x_2 = -1, x_3 = 2$

Example:- Determine the values of a and b for which the system

$x+2y+3z=6$

$x+3y+5z=9$

$2x+5y+az=6$

has (i) no solution (ii) Unique solution (iii) infinite number of solutions

Sol:-

$$[A/B] = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 & 9 \\ 2 & 5 & a & b \end{bmatrix}$$

$$\begin{matrix} R_{21}(-1) \\ R_{31}(-2) \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & a-6 & b-12 \end{bmatrix}$$

$$R_{32}(-1) \sim \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a-8 & b-15 \end{bmatrix}$$

Case 1:- if $a=8,\ b\neq 15$, $r(A)=2\neq 3=r(A|B)$. inconsistent

∴ System has no solutions

Case 2:- if $a\neq 8$, b any value, $r(A)=3=r(A|B)=n=$ no of variables,

unique solution, $z=\frac{b-15}{a-8}$

$y+2z=3 \Rightarrow y+2\frac{(b-15)}{(a-8)}=3$

$\Rightarrow y = 3-\frac{2(b-15)}{a-8}$

$=\frac{3a-24-2b+30}{a-8}$

$=\frac{3a-2b+6}{a-8}$

$x+2y+3z=6 \Rightarrow x+2\left(\frac{3a-2b+6}{a-8}\right)+3\left(\frac{b-15}{a-8}\right)=6$

$\Rightarrow x = 6-\frac{2(3a-2b+6)+3(b-15)}{a-8}$

$=\frac{6a-48-6a+4b-12-3b+45}{a-8}$

$=\frac{b-15}{a-8}$

∴ $x=z=\frac{b-15}{a-8}$

Case 3: If $a=8, b=15$, $r(A)=2=r(A|B)<3=n$, infinite solutions with $n-r=3-2=1$ arbitrary variable constants.

let $z=k$, $y+2z=3 \Rightarrow y=3-2k$

$x+2y+3z=6 \Rightarrow x+2(3-2k)+3k=6$

$\Rightarrow x+6-4k+3k=6$

$\Rightarrow x=k$.

BV Ramana

II Find the values of a & b for which the system has (i) no solution (ii) unique sol (iii) infinitely many sol for

$2x+3y+5z=9$

$7x+3y-2z=8$,

$2x+3y+az=b$.

Sol:

$$[A|B]=\begin{bmatrix}2&3&5&9\\7&3&-2&8\\2&3&a&b\end{bmatrix} \underset{R_{31}(-1)}{R_{21}(-7)} \begin{bmatrix}1&3&5&9\\0&-18&-37&-55\\0&-3&a-10&b-18\end{bmatrix} R_{32}(-\tfrac{1}{6}) \sim \begin{bmatrix}1&3&5&9\\0&-18&-37\end{bmatrix}$$

$$R_{32}(-\tfrac{1}{6}) \sim \begin{matrix}1&3&5&9\\0&-18&-37&-55\\0&0&\frac{6a-23}{6}&\frac{6b-53}{6}\end{matrix}$$

$$R_{12} \sim \begin{bmatrix}7&3&-2&8\\2&3&5&9\\2&3&a&b\end{bmatrix} \underset{R_{23}(-1)}{R_{12}(-3)} \sim \begin{matrix}1&-6&-17&-10\\2&3&5&9\\0&0&a-5&b-9\end{matrix}$$

$$R_{21}(-3) \begin{bmatrix}2&3&5&9\\1&-6&-17&-10\\2&3&a&b\end{bmatrix} \underset{R_{31}(-1)}{R_{21}(-1/2)} \begin{bmatrix}2&3&5&9\\0&-\frac{15}{2}&-\frac{39}{2}&-\frac{29}{2}\\0&0&a-5&b-9\end{bmatrix}$$

Case(i) If $a=5$, $b\neq 9$ then system has no solution

Case(ii) If $a\neq 3$, b any value system has unique sol

$$z=\frac{b-9}{a-5} \qquad -\frac{15}{2}y-\frac{39}{2}z=-\frac{29}{2}$$

$$y=\frac{29a-39b+206}{15(a-5)} \qquad \Rightarrow -15y-39z=-29$$

$$\Rightarrow -15y-39\left(\frac{b-9}{a-5}\right)=-29$$

$$\Rightarrow -15y=-29+39\frac{(b-9)}{a-5}$$

Sol 12.

$$A|B = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & a & b \end{bmatrix} \xrightarrow{R_{21}(-1),\ R_{31}(-1)} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & a-1 & b-6 \end{bmatrix} \xrightarrow{R_{32}(-1)} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & a-3 & b-10 \end{bmatrix}$$

Case 1:- if $a = 3$, $b \neq 10$, System is inconsistent.

Case 2:- if $a \neq 3$, b is any value system has unique sol

$z = \frac{b-10}{a-3}$, $y + 2z = 4$

$y + 2\left(\frac{b-10}{a-3}\right) = 4$

$\Rightarrow y = 4 - \frac{2b-20}{a-3} = \frac{4a-12-2b+20}{a-3} = \frac{4a-2b+8}{a-3}$

and $x + y + z = 6$

$\Rightarrow x + \frac{4a-2b+8}{a-3} + \frac{b-10}{a-3} = 6$

$\Rightarrow x = 6 - \frac{b-10}{a-3} - \frac{4a-2b+8}{a-3}$

$= \frac{6a-18-b+10-4a+2b-8}{a-3}$

$= \frac{2a+b-16}{a-3}$

Case 3:- If $a = 3$, $b = 10$ then system is consistent and infinitely many solutions exists $n - r = 3 - 2 = 1$ arbitrary variable k

let $z = k$

i.e. $y + 2z = 4$

$\Rightarrow y + 2k = 4 \Rightarrow y = 4 - 2k$

and $x + y + z = 6 \Rightarrow x + 4 - 2k + k = 6$

$\Rightarrow x = 2 + k$

$\therefore x = 2 + k,\ y = 4 - 2k,\ z = k$

Solution of Homogeneous System of Equations

Procedure:

Let rank of $A = r$ and rank of $A|B = r_1$

Since all b's are zero, $r = r_1$, then

Step (i): If $r = n$ = no of variables

Then the system of equations have only trivial solution

(ii) If $r < n$ = number of unknowns/variables

Then the system of equations have an infinite number of non-trivial solutions, we shall have $n-r$ linearly independent solutions.

Note: If A is a non-singular matrix (i.e., $\det A \neq 0$) then the system $Ax = 0$ has only trivial solution

Note If A is a singular matrix (i.e., $\det A = 0$) then the system posses a non-zero solution.

System of Homogeneous Equations:-

Result:- If $r<m$, omit $m-r$ equations such that the coefficient matrix of the remaining equations still has rank r. Rewrite r unknowns in terms of $n-r$ arbitrary unknowns and solve.

Result 2:- If $m<n$, system has non-trivial solutions.
m is no of equations and n is no of unknowns/variables

Result 3:- If $m=n$, system has non-trivial solution if it's coefficient determinent is zero.

Result 4:- A homogeneous system always has a trivial solution

Since $r[A|B] = r[A|0] = r(A) = n$

Solve the system of homogeneous equations:-

1. $x+2y+3z=0,$
$3x+4y+4z=0,$
$7x+10y+12z=0.$

Sol:- The coefficient matrix of A is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix} \underset{R_{31}(-7)}{R_{21}(-3)} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix} R_{32}(-2) \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2(-\tfrac{1}{2}) \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5/2 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ The rank of $A = 3 = n =$ no of variables.

i.e. $|A| \neq 0$. System has only trivial solution.

$\therefore$ $x=0, y=0,$ and $z=0$

2. $4x + 2y + z + 3w = 0$

$6x + 3y + 4z + 7w = 0$

$2x + y + w = 0$

. JNTU - 2004, 2005, 2006S

Sol: The coefficient matrix A is

$$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix} \sim_{\substack{R_{21}(-3/2) \\ R_{31}(-1/2)}} \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & +5/2 & 5/2 \\ 0 & 0 & -1/2 & -\frac{1}{2} \end{bmatrix}$$

$$R_{32}(1/5) \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5/2 & 5/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}_r$$

$\therefore \gamma(A) = \gamma(A/B) = 2 < 4 = n =$ number of variables.

So, Nontrivial solution exist in terms of $n - r = 4 - 2 = 2$ variables

Choose $x = K_1$, and $w = K_2$. then solving.

$$4x + 2y + z + 3w = 0$$

$$5/2\, z + 5/2\, w = 0$$

$$z = -w = -K_2$$

and $4k_1 + 2y - k_2 + 3k_2 = 0 \Rightarrow 4x_1 + 2K_1 - K_2 + 3K_2 = 0$

$4x_1 = -2K_1 - 2K_2$

$\Rightarrow 4k_1 + 2y = 0$

$x_1 = -\frac{1}{2}[K_1 + K_2]$

$\Rightarrow y = -2K_1$

Where K_1 and K_2 are arbitrary constants, giving infinite number of solutions.

$\therefore x = K_1$; $y = -2K_1$, $z = -K_2$, and $w = K_2$

Assignment problems:-

(1) Solve the system of equations

$x + 3y - 2z = 0; \quad 2x - y + 4z = 0; \quad x - 11y + 14z = 0$ [JNTU 2002]

(2) Find the values of λ for which the equations

$(\lambda-1)x + (3\lambda+1)y + 2\lambda z = 0$ JNTU 2004

$(\lambda-1)x + (4\lambda-2)y + (\lambda+3)z = 0$

$2x + (3\lambda+1)y + 3(\lambda-1)z = 0$

are consistent and find the ratio of $x:y:z$ when λ has the smallest of these values. What happens when λ has the greater of these values

(3) ~~$4x+2y+z+3w=0$, $6x+3y+4z$~~

$3x + 4y - z - 6w = 0; \quad 2x + 3y + 2z - 3w = 0; \quad 2x + y - 14z - 9w = 0;$

$x + 3y + 13z + 8w = 0$ [JNTU 2002]

(4) $x + y - z + t = 0; \quad x - 2y + 2z - t = 0; \quad 3x + y + t = 0$

(5) $x + 3y + 2z = 0; \quad 2x - y + 3z = 0; \quad 3x - 5y + 4z = 0; \quad x + 17y + 4z = 0$

(6) Solve $x_1 + 2x_3 - 2x_4 = 0; \quad 2x_1 - x_2 - x_4 = 0; \quad x_1 + 2x_3 - x_4 = 0; \quad 4x_1 - x_2 + 3x_3 - x_4 = 0$

(7) 2006 $x + y - 3z + 2w = 0; \quad 2x - y + 2z - 3w = 0, \quad 3x - 2y + z - 4w = 0; \quad -4x + y - 3z + w = 0$

(8) 2006 $x + y - 2z + 3w = 0; \quad x - 2y + z - w = 0; \quad 4x + y - 5z + 8w = 0; \quad 5x - 7y + 2z - w = 0$

(9) Determine b such that the system of homogeneous equations.

$2x + y + 2z = 0$

(10) $x_1 + x_2 + x_3 + x_4 = 0, \quad x_1 + 3x_2 + 2x_3 + 4x_4 = 0, \quad 2x_1 + x_3 - x_4 = 0;$

(11) 2007 $3x_1 + x_2 - \lambda x_3 = 0, \quad 4x_1 - 2x_2 - 3x_3 = 0, \quad 2\lambda x_1 + 4x_2 + \lambda x_3 = 0$

(12) $4x + 2y + z + 3w = 0, \quad 6x + 3y + 4z + 7w = 0,$

2005, 2002, 2006,

(13) Show that the only real ~~and~~ value of λ for which the following equations have non-trivial sol is 6 and solve them

$\lambda = 6, \quad x + 2y + 3z = \lambda x, \quad 3x + y + 2z = \lambda y, \quad 2x + 3y + z = \lambda z$ $\quad |A| = \lambda^3 - 3\lambda^2 - 15\lambda + 18 = 0$

Om Sai Ram

Solutions of linear Systems - Direct Methods

1. Solve the following system by Gaussian elimination method

$$2x_1 - 7x_2 + 4x_3 = 9$$
$$x_1 + 9x_2 - 6x_3 = 1$$
$$-3x_1 + 8x_2 + 5x_3 = 6$$

Sol:- The augmented matrix is $[A|B] = \begin{bmatrix} 2 & -7 & 4 & 9 \\ 1 & 9 & -6 & 1 \\ -3 & 8 & 5 & 6 \end{bmatrix}$

$$\underset{R_{31}(3/2)}{\overset{R_{21}(-1/2)}{\sim}} \begin{bmatrix} 2 & -7 & 4 & 9 \\ 0 & \frac{25}{2} & -8 & -7/2 \\ 0 & -5/2 & 11 & 39/2 \end{bmatrix}$$

$$\underset{R_{32}(-\frac{1}{5})}{\sim} \begin{bmatrix} 2 & -7 & 4 & 9 \\ 0 & \frac{25}{2} & -8 & -7/2 \\ 0 & 0 & \frac{47}{5} & \frac{94}{5} \end{bmatrix}$$

This corresponds to the upper triangular System

By back Sub

$$2x_1 - 7x_2 + 4x_3 = 9$$
$$\frac{25}{2}x_2 - 8x_3 = -7/2$$
$$\frac{47}{5}x_3 = \frac{94}{5} \Rightarrow x_3 = 2$$

$$\frac{25}{2}x_2 - 16 = -7/2 \Rightarrow 25/2\, x_2 = 16 - 7/2$$
$$\frac{25}{2}x_2 = \frac{32-7}{2} = \frac{25}{2}$$
$$\therefore x_2 = 1$$

$$2x_1 - 7 + 8 = 9$$
$$\Rightarrow 2x_1 + 1 = 9 \Rightarrow 2x_1 = 8 \Rightarrow x_1 = 4$$
$$\therefore x_1 = 4,\ x_2 = 1,\ x_3 = 2$$

② Solve the equations $x+y+z=6$; $3x+3y+4z=20$; $2x+y+3z=13$ using Gaussian elimination method $\underline{x=3}$, $\underline{y=1}$, $\underline{z=2}$

③ Solve the system of equations $3x+y-z=3$; $2x-8y+z=-5$; $x-2y+9z=8$ using G-E·M $z=1, y=1, x=1$

④ Solve the system of eqs $2x_1+x_2+x_3=10$, $3x_1+2x_2+3x_3=18$, $x_1+4x_2+9x_3=16$

$x_3=5, x_2=-9, x_1=7$

⑤
$$2x_1+2x_2+x_3+2x_4=7$$
$$-x_1+2x_2+x_4=-2$$
$$-3x_1+x_2+2x_3+x_4=-3$$
$$-x_1+2x_4=0$$

$x_4=1.08$, $x_3=1.4324$, $x_2=-0.4562$, $x_1=2.1600$

⑥
$$2x_1+5x_2+2x_3-3x_4=3$$
$$3x_1+6x_2+5x_3+2x_4=2$$
$$4x_1+5x_2+14x_3+14x_4=11$$
$$5x_1+10x_2+8x_3+4x_4=4$$

$x_1=-66, x_2=27, x_3=6, x_4=4$

Solutions of System of non-Homogeneous equations

Method of Factorization [Triangularization or Decomposition]

This method is based on the fact that square matrix A can be factorized into the form LU where L is the lower triangular matrix and U is the upper triangular matrix. Here all the principle minors of A must be non-singular. This factorization, if it exists, is unique.

Consider the linear system

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$ which can be written as

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

$$AX = B \quad -(1)$$

let $A = LU \quad -(2)$ where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$, and $U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$.

Here $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ = Coefficient matrix of of the linear system (eq)

L is the lower triangular matrix

U is the unit upper triangular matrix

This method is known as Crout's Method.

After finding L and U substitute (2) in (1)

i.e $LUX = B \quad -(3)$ Now take $UX = Y. \quad -(4)$

$\Rightarrow LY = B$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

After finding Y sub in (4) then we get X.

Problems:- Using Crouts method, solve the following system of equations $x_1 + x_2 + x_3 = 1$, $4x_1 + 3x_2 - x_3 = 6$, $3x_1 + 5x_2 + 3x_3 =$

Sol. Given system of equations can be written as $AX = B$ —(1)

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ & $B = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$

Let $A = LU$ —(2) where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ & $U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

$$LU = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

$l_{11} = 1$ $\qquad$ $l_{11}u_{12} = 1 \Rightarrow u_{12} = 1$ $\qquad$ $l_{11}u_{13} = 1$, $u_{13} = 1$

$l_{21} = 4$ $\qquad$ $l_{21}u_{12} + l_{22} = 3$
$\Rightarrow 4 \cdot 1 + l_{22} = 3$
$\Rightarrow l_{22} = -1$

$l_{21}u_{13} + l_{22}u_{23} = -1$
$\Rightarrow 4 \cdot (1) + (-1) \cdot u_{23} = -1$
$\Rightarrow -u_{23} = -5 \Rightarrow u_{23} = 5$

$l_{31} = 3$ $\qquad$ $l_{31}u_{12} + l_{32} = 5$
$\Rightarrow 3(1) + l_{32} = 5$
$\Rightarrow l_{32} = 2$

$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 3$
$\Rightarrow 3(1) + 2(5) + l_{33} = 3$
$\Rightarrow l_{33} = -10$

$$L = \begin{bmatrix} L_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

Now substitute (2) in (1) $\Rightarrow$ $LUX = B \rightarrow$ (3)

eq (3) is

Consider $UX = Y$ Then $LY = B$ — (5)

— (4)

where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

from (5) $LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix} \Rightarrow$$

$y_1 = 1$

$4y_1 - y_2 = 6 \Rightarrow y_2 = -2$

$3y_1 + 2y_2 - 10y_3 = 4$

$\Rightarrow 3 - 4 - 10y_3 = 4$

$\Rightarrow -10y_3 = 1 + 4$

$\Rightarrow y_3 = -5/10 = -1/2$

$$\therefore Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1/2 \end{bmatrix}$$

Sub Y in (4) Then $UX = Y$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1/2 \end{bmatrix}$$

$\Rightarrow x_1 + x_2 + x_3 = 1 \qquad x_1 + 1/2 - 1/2 = 1 \Rightarrow x_1 = 1$

$x_2 + 5x_3 = -2 \Rightarrow x_2 - 5/2 = -2 \Rightarrow x_2 = -2 + 5/2 = \frac{1}{2}$

$x_3 = -1/2$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ -1/2 \end{bmatrix}$$

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

Now substitute (2) in (1) $\Rightarrow$ $LUX = B \rightarrow$ (3)

eq (3) is

Consider $UX = Y$ Then $LY = B$ — (5)

— (4)

where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

from (5) $LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix} \Rightarrow$$

$y_1 = 1$

$4y_1 - y_2 = 6 \Rightarrow y_2 = -2$

$3y_1 + 2y_2 - 10y_3 = 4$

$\Rightarrow 3 - 4 - 10y_3 = 4$

$\Rightarrow -10y_3 = 1 + 5$

$\Rightarrow y_3 = -5/10 = -1/2$

$$\therefore Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1/2 \end{bmatrix}$$

Sub Y in (4) Then $UX = Y$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1/2 \end{bmatrix}$$

$\Rightarrow x_1 + x_2 + x_3 = 1 \quad x_1 + 1/2 - 1/2 = 1 \Rightarrow x_1 = 1$

$x_2 + 5x_3 = -2 \Rightarrow x_2 - 5/2 = -2 \Rightarrow x_2 = -2 + 5/2 = \frac{1}{2}$

$x_3 = -1/2$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ -1/2 \end{bmatrix}$$

i.e $A(c_1x_1 + c_2x_2) = c_1Ax_1 + c_2Ax_2$

$= c_1y_1 + c_2y_2$

where c_1 and c_2 are constants. Inverse transformation of ② is $X = A^{-1}Y$

Def: let A be $n \times n$ square matrix. Suppose there exists a non-zero column vector x of order n and a real or complex number λ such that

$$AX = \lambda X$$

[i.e., Suppose the linear transforms $Y = Ax$ transform x into a scalar multiple of itself]

Then the unknown scalar 'λ' is known as an eigen value of the matrix 'A' and the corresponding non-zero vector x as eigen vector. Thus eigen values or characteristic values or latent or proper values are scalars 'λ' which satisfy the equation

$$AX = \lambda X \quad — (3)$$

for any $X \neq 0$

$$\Rightarrow (AX - \lambda X) = 0$$

$$\Rightarrow (A - \lambda I)X = 0 \quad — (4)$$

Equation (4) represents a system of n-homogeneous equations in the n-variables $x_1, x_2, \ldots, x_n$, (4) has non-trivial solution if the coefficient matrix $(A - \lambda I)$ is singular

i.e., $|A - \lambda I| = 0 \quad — (5)$

(or) $$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \quad - (6)$$

Expanision of the determinent gives a n^{th} degree polynomial $P_n(x)$ known as the characteristic polynomial of A.

Eq (5) is known as characteristic equation of

Thus eigen values of n-square matrix A are the roots of the characteristic equation (6). Hence 'A' can have atleast one and atmost 'n' - eigen values.

Degree of the characteristic equation = Order of matrix A

Spectrum of A is the set of all eigen values of A

Procedure to obtain Eigen Values and Eigen Vectors:-

→ Solve the Characteristic equation $|A-\lambda I| = 0$

For eigen values λ_i. If A is of n^{th} order, the number of eigen values are n or lessthan 'n' [with repeated real roots or complex conjugate pairs]

→ For a specific eigen value λ_i, Solve the ~~homog~~ homogeneous system of equations $(A-\lambda_i I)X = 0$

Properties:

1 → Trace: of A

Trace of a matrix A is the sum of diagonal elements of A (or) sum of the roots of the characteristic polynomial (or) $\lambda_1 + \lambda_2 + \cdots + \lambda_n$ and is denoted by σ_1

i.e., Trace (σ_1) = sum of the eigen values of A

2 → Determinant of A is product of principal diagonal elements of A (or) product of eigen values of A

i.e., $\det A = |A| = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$

3 → Singular :- If atleast one eigen value is zero then $|A| = 0$.

since $|A| = \lambda_1 \cdot \lambda_2 \cdots \lambda_n = 0$.

i.e, A is singular

4 → Two vectors X and Y are said to be orthogonal if $X^T Y = Y^T X = 0$

Properties of Eigen Values and Eigen Vectors.

Property 1:-

The sum of the eigen values of a square matrix is equal to it's trace and product of the eigen values is equal to it's determinent.

i.e if A is an $n\times n$ matrix and $\lambda_1, \lambda_2, \dots, \lambda_n$ are it's n- eigen values $\lambda_1+\lambda_2+\dots+\lambda_n = Tr(A)$ and $\lambda_1 \cdot \lambda_2 \cdots \lambda_n = \det A$

Proof: characteristic equation of A is $|A-\lambda I|=0$

i.e $\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots \\ \cdots & \cdots & & \cdots \end{vmatrix} = 0$

i.e $(a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)\dots(a_{nn}-\lambda)$ + a polynomial of degree $(n-2)$ + ... $= 0$

$\Rightarrow (-1)^n(\lambda-a_{11})(\lambda-a_{22})\dots(\lambda-a_{nn})$ + a polynomial of degree $(n-2)$ + ... $= 0$

$\Rightarrow (-1)^n[\lambda^n - (a_{11}+a_{22}+\dots+a_{nn})\lambda^{n-1}$ + a polynomial of degree $(n-2)$ + ...] + a polynomial of degree $(n-2) = 0$

$\Rightarrow (-1)^n\lambda^n + (-1)^{n+1}(a_{11}+a_{22}+\dots+a_{nn})\lambda^{n-1}$ + a polynomial of degree $(n-2)$ + ... $= 0$ —①

let $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ be the roots of ①, which are nothing but eigen values of A

Now sum of the roots $= \lambda_1+\lambda_2+\dots+\lambda_n$

$= -\dfrac{(-1)^{n+1}(a_{11}+a_{22}+\dots+a_{nn})}{(-1)^n}$

$= a_{11}+a_{22}+\dots+a_{nn}$ = Trace of A

* Since Sum of the roots of $ax^2+bx+c=0$ is $-b/a$

$\therefore$ Sum of eigen values of A = Trace of A

(ii) In general the characteristic equation of A of order n, will be of the form

$|A-\lambda I| = (-1)^n \lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \dots + k_n\lambda^0 = 0$ — (2)

put $\lambda = 0$

$\Rightarrow |A| = k_n$

$\Rightarrow k_n =$ determinent of A — (3)

let $\lambda_1, \lambda_2 \dots, \lambda_n$ be the roots of (2) which will become the eigen values of A.

$\therefore$ Product of the roots $= \lambda_1 \cdot \lambda_2 \cdots \lambda_n$

$= \dfrac{(-1)^n k_n}{(-1)^n}$

$= k_n = |A|$ from (3)

Product of eigen values $= |A|$

Property 2. The eigen values of a diagonal matrix are nothing but its principal diagonal elements.

Property 3:- Eigen values of a triangular matrix are nothing but its principal diagonal elements.

Property 4:- The eigen values of a matrix A and its transpose A^T are same

proof: We have $(A-\lambda I)^T = A^T - (\lambda I)^T = A^T - \lambda (I)^T$

$= A^T - \lambda I$

Also we know that $|(A-\lambda I)^T| = |A-\lambda I|$

$\Rightarrow |A^T - \lambda I| = |A-\lambda I|$ from (1)

i.e $|A-\lambda I| = 0 \Leftrightarrow |A^T - \lambda I| = 0$

i.e roots of $|A-\lambda I| = 0$ and $|A^T - \lambda I| = 0$ are same

i.e The eigen values of A and A^T are same

Property 5: If λ is an eigen value of A which is invertible then $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

proof: Given λ is the eigen value of A

let x be the eigen vector corresponding to the eigen value λ.

$$\therefore AX = \lambda X$$

multiply by A^{-1}

$$\Rightarrow A^{-1}(AX) = A^{-1}(\lambda X)$$

$$\Rightarrow (A^{-1}A)X = (A^{-1}\lambda)X$$

$$\Rightarrow IX = \lambda(A^{-1}X)$$

$$\Rightarrow \lambda(A^{-1}X) = X$$

$$\Rightarrow A^{-1}X = \frac{1}{\lambda}X$$

$\therefore \frac{1}{\lambda}$ is the eigen value of A^{-1} and x is the corresponding eigen vector.

Note: In general if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of a square matrix A which is invertable then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2} \ldots \frac{1}{\lambda_n}$ are eigen values of A^{-1}

(no need give proof)

Property 6: If λ is an eigen value of A and k is a non-zero scalar, then $k\lambda$ is an eigen value of kA

proof: Given λ is an eigen value of A

let x be the eigen vector of A corresponding to λ

$\therefore AX = \lambda X \Rightarrow k(AX) = k(\lambda X) \Rightarrow (kA)X = (k\lambda)X \Rightarrow$

$k\lambda$ is an eigen value of kA

Note: If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of a square matrix A of order n and if k is a non-zero scalar then $k\lambda_1, k\lambda_2, k\lambda_3, \ldots, k\lambda_n$ are the eigen values of kA.

Property 7:- If λ is eigen value of A then λ^n is an eigen value of A^n where n is +ve integer.

proof: Given λ is a eigen value of A

Let X be the corresponding eigen vector

$$AX = \lambda X \quad -(1)$$

$$\Rightarrow A(AX) = A(\lambda X) \Rightarrow A^2X = \lambda(AX)$$

$$\Rightarrow A^2X = \lambda(\lambda X) \cdot \text{ from (1)}$$

$$\Rightarrow A^2X = \lambda^2 X$$

$\therefore \lambda^2$ is an eigen value of A^2

i.e., the result is true when $n=2$

Now let the result be true when $n=k$

i.e., λ^k is an eigen value of A^k

$$\therefore A^k X = \lambda^k X$$

$$\Rightarrow A(A^k X) = A(\lambda^k X)$$

$$\Rightarrow A^{k+1}X = \lambda^k (AX)$$

$$\Rightarrow A^{k+1}X = \lambda^k (\lambda X) \quad \text{from (1)}$$

$$\Rightarrow A^{k+1}X = \lambda^{k+1} X$$

$\therefore \lambda^{k+1}$ is an eigen value of A^{k+1}

$\therefore$ By the principle of mathematical induction, we have λ^n is the eigen value of A^n for any +ve integer.

Property 8: If λ is an eigen value of A, then $\lambda+k$ is an eigen value of $A+kI$, where k is a non zero scalar

Property 9: If λ is an eigen value of A, then $a_0\lambda^2+a_1\lambda+a_2$ is an eigen value of $a_0A^2+a_1A+a_2$

Note: If λ is an eigen value of ~~$a_0A^2+a_1A+a_2$~~ A, then $a_0\lambda^n+a_1\lambda^{n-1}+a_2\lambda^{n-2}+\cdots+a_n$ is an eigen value of $a_0A^n+a_1A^{n-1}+a_2A^{n-2}+\cdots+a_nI$.

Property 10: If A and P are two square matrices of same order and if P is invertable, then A and $P^{-1}AP$ will have same eigen values

Proof: Consider $|P^{-1}AP - \lambda I| = |P^{-1}AP - \lambda P^{-1}P|$

$$= |(P^{-1}A - \lambda P^{-1})P| = |P^{-1}A - \lambda P^{-1}|\,|P|$$

$$= |P^{-1}(A - \lambda I)|\,|P| = |P^{-1}|\,|A - \lambda I|\,|P|$$

$$= |P^{-1}|\,|P|\,|A - \lambda I|$$

$$= |P^{-1}P|\,|A - \lambda I| = |I|\,|A - \lambda I|$$

$$= |A - \lambda I|$$

$$\therefore |P^{-1}AP - \lambda I| = |A - \lambda I|$$

i.e., $|P^{-1}AP - \lambda I| = 0$ if and only if $|A - \lambda I| = 0$

$\therefore$ The characteristic equation of $P^{-1}AP$ and A are one and the same

i.e., The roots of the equations are same

i.e., The roots of $P^{-1}AP$ and A are same.

Hence the eigen values of $P^{-1}AP$ and A are same.

Corollary:- If A and B are square matrices such that A is non-singular, then $A^{-1}B$ and $B^{-1}A$ have the same eigen values.

proof: In the previous proof take BA^{-1} in place of A and A in place of P.

We deduce that $A^{-1}(BA^{-1})A$ and BA^{-1} have the same eigen values.

i.e., $(A^{-1}B)(A^{-1}A)$ and BA^{-1} have same eigen values

i.e., $(A^{-1}B)(I)$ and BA^{-1} have same eigen values i.e., $A^{-1}B$ and BA^{-1} have the same eigen values.

Corrolory: If A and B are square matrices of same order which are invertable, then AB and BA will have same eigen values.

Property: 11:— If λ is an eigen value of A, then $\frac{|A|}{\lambda}$ is an eigen value of adj A

proof: Let X be the eigen vector of A corresponding to the eigen value of λ.

$$AX = \lambda X \quad — ①$$

Now consider $(\text{adj} A) AX = (\text{adj} A) \lambda X$ from ①

$$\Rightarrow |A| I \cdot X = \lambda (\text{adj} A) X$$

$$\therefore (\text{adj} A) X = \frac{|A|}{\lambda} \cdot X.$$

$\therefore \frac{|A|}{\lambda}$ is an eigen value of adj A.

Property 12:- If λ is an eigen value of an orthogonal matrix A, then $\frac{1}{\lambda}$ is also an eigen value of A.

Property 13:- An eigen vector of A does not correspond to more than one eigen value

proof: If possible, let X be the eigen vector of A corresponding to two distinct eigen values λ_1 & λ_2 i.e. $(\lambda_1 \neq \lambda_2)$

$$\therefore AX = \lambda_1 X \rightarrow ① \text{ and } AX = \lambda_2 X \rightarrow ②$$

from ① and ② $\lambda_1 X = \lambda_2 X \Rightarrow \lambda_1 X - \lambda_2 X = 0$

$$\Rightarrow (\lambda_1 - \lambda_2) X = 0 \Rightarrow \lambda_1 - \lambda_2 = 0 \quad (\because X \neq 0)$$

$$\Rightarrow \lambda_1 = \lambda_2$$

which is a contradiction to the fact that λ_1 & λ_2 are distinct

$\therefore$ Our assumption is wrong

$\therefore$ An eigen vector doesn't correspond to more than one eigen value of a matrix.

Problems

1. Find the eigen value and the eigen vecter of

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

Sol: The eigen values are nothing but the roots of the equation $|A-\lambda I|=0$

i.e., $\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$

$\Rightarrow (-5-\lambda)(-2-\lambda) - 4 = 0$

$\Rightarrow 10 + 7\lambda + \lambda^2 - 4 = 0$

$\Rightarrow \lambda^2 + 7\lambda + 6 = 0 \Rightarrow \lambda^2 + 6\lambda + \lambda + 6 = 0$

$\Rightarrow \lambda(\lambda+6) + 1(\lambda+6) = 0 \Rightarrow (\lambda+1)(\lambda+6) = 0$

$\Rightarrow \lambda = -1, -6$

i.e., $\lambda_1 = -1$ and $\lambda_2 = -6$

Eigen vector corresponding to eigen value $\lambda = -1$

i.e., $(A-\lambda I)X = 0$

$\Rightarrow \begin{bmatrix} -5+1 & 2 \\ 2 & -2+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \left.\begin{matrix} -4x_1 + 2x_2 = 0 \\ 2x_1 - x_2 = 0 \end{matrix}\right\} \Rightarrow 2x_1 = x_2$

choose $x_1 = k$, then $x_2 = 2k$

$\therefore e_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ 2k \end{bmatrix} = k \begin{bmatrix} 1 \\ 2 \end{bmatrix} = k \cdot X_1$

i.e., $X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Eigen Vector Corresponding $\lambda_2 = -6$:-

$$(A - \lambda I)X = 0$$

$$\Rightarrow \begin{bmatrix} -5+6 & 2 \\ 2 & -2+6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{matrix} x_1 + 2x_2 = 0 \\ 2x_1 + 4x_2 = 0 \end{matrix}\right\} \Rightarrow x_1 = -2x_2$$

If $x_2 = k$ then $x_1 = -2k$

$$\therefore e_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2k \\ k \end{bmatrix} = k \begin{bmatrix} -2 \\ 1 \end{bmatrix} = k X_2$$

i.e., $X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\therefore$ For the eigen values $\lambda_1 = -1$ and $\lambda_2 = -6$

corresponding eigen vectors are

$$X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

2. Find the eigen value of the given matrix and find the eigen vectors.

(i) $A = \begin{bmatrix} 6 & 8 \\ 8 & -6 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

3. Find the eigen vectors and eigen value of a given matrix

(i) $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Sol: The characteristic equation of A is $|A - \lambda I| = 0$

i.e $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

$$\Rightarrow (-2-\lambda)[-\lambda(1-\lambda) - 12] - 2[-2\lambda - 6] + (-1)[-12 + 3(1-\lambda)] = 0$$

$\Rightarrow (-2-\lambda)[-\lambda+\lambda^2-12]+4\lambda+12-[-12+3-3\lambda]=0$

$\Rightarrow (-2-\lambda)[\lambda^2-\lambda-12]+4\lambda+12-(-3\lambda-15)=0$

$\Rightarrow -2\lambda^2+2\lambda+24-\lambda^3+\lambda^2+12\lambda+4\lambda+12+3\lambda+15=0$

$\Rightarrow -\lambda^3-\lambda^2+21\lambda+45=0$

$\Rightarrow \lambda^3+\lambda^2-21\lambda-45=0$

$\Rightarrow (\lambda-5)(\lambda+3)^2=0$

$\Rightarrow \lambda=5,-3,-3$

5	1	1	-21	-45
	0	5	30	45
	1	6	9	(0)

$(\lambda-5)(\lambda^2+6\lambda+9)=0$

$\Rightarrow (\lambda-5)(\lambda+3)(\lambda+3)=0$

Eigen vector corresponding $\lambda=5$

i.e., $(A-\lambda I)X=0$

$$\Rightarrow \begin{bmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -7x_1+2x_2-3x_3=0$ — (1)

$2x_1-4x_2-6x_3=0$ — (2)

$-x_1-2x_2-5x_3=0$ — (3)

from (2) and (3)

(2)+(3)×2 $\Rightarrow$ $2x_1-4x_2-6x_3=0$

$-2x_1-4x_2-10x_3=0$

$-8x_2-16x_3=0$

$\Rightarrow 8x_2=-16x_3$

$\Rightarrow x_2=-2x_3$ Choose $x_3=k$

$\Rightarrow x_2=-2k$

substitute x_3 and x_2 in (1)

$-7x_1 + 2(-2k) - 3k = 0$

$\Rightarrow -7x_1 - 7k = 0 \Rightarrow x_1 = -k$

$$\therefore e_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k \\ -2k \\ k \end{bmatrix} = k\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} = k \cdot X_1$$

i.e., $X_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

<u>For $\lambda = -3$</u>

i.e $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} -2+3 & 2 & -3 \\ 2 & 1+3 & -6 \\ -1 & -2 & +3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 + 4x_2 - 6x_3 &= 0 \\ -x_1 - 2x_2 + 3x_3 &= 0 \end{aligned}\right\}$$ There is no posible to solutions by using equation solving method

$$\begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & +0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 + 2x_2 - 3x_3 = 0 \quad (1)$$

Then the rank of A is 1 and the no of variables are 3. Then choose $n - r = 3 - 1 = 2$ arbitrary variables

i.e $x_2 = k_1$ and $x_3 = k_2$ sub in (1)

$\Rightarrow x_1 + 2k_1 - 3k_2 = 0 \Rightarrow x_1 = -2k_1 + 3k_2$

$$\therefore e_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2k_1 + 3k_2 \\ k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -2k_1 \\ k_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3k_2 \\ 0 \\ k_2 \end{bmatrix}$$

$$= k_1\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + k_2\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = k_1 X_2 + k_2 X_3$$

$\therefore \; X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Thus the eigen values of A is are $\lambda_1 = 5$, $\lambda_2 = -3$ & $\lambda_3 = -$

corresponding eigen vectors are.

$$X_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \; X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

(ii) $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$. (iii) $A = \begin{bmatrix} 3 & -2 & -5 \\ 4 & -1 & -5 \\ -2 & -1 & -3 \end{bmatrix}$

(iv) Find the eigen vectors and eigen values of

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

Solution:- The characteristic equation $|A - \lambda I| = 0$

i.e $\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$

$\Rightarrow (3-\lambda)(2-\lambda)(5-\lambda) = 0 \Rightarrow \lambda = 2, 3, 5$

Eigen vector for $\lambda = 3$

i.e $(A - \lambda I)X = 0$

$\Rightarrow (A - 3I)X = 0$

$\Rightarrow \begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \left.\begin{array}{l} x_2 + 4x_3 = 0 \\ x_2 + 6x_3 = 0 \\ 2x_3 = 0 \end{array}\right\} \Rightarrow x_2 = -4x_3 = 0$

$\Rightarrow x_3 = 0$

Choose $x_1 = k$.

Then $e_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = kX_1$

$\therefore X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Eigen vector for $\lambda = 2$

we know that $(A - \lambda I)X = 0$

i.e., $(A - 2I)X = 0$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow x_1 + x_2 + 4x_3 = 0$ — ①

$\left. \begin{array}{l} 6x_3 = 0 \\ 3x_3 = 0 \end{array} \right\} x_3 = 0$

from ① $\Rightarrow x_1 + x_2 = 0$ choose $x_2 = k$.

$\Rightarrow x_1 = -x_2$ Then $x_1 = -k$

i.e., $e_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = kX_2$

$\therefore X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Eigen vector for $\lambda = 3$

we have $(A - \lambda I)X = 0$

$\Rightarrow (A - 3I)X = 0$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -2x_1 + x_2 + 4x_3 = 0$ — ①

$-3x_2 + 6x_3 = 0 \Rightarrow x_2 = 2x_3$

If $x_3 = k$ then $x_2 = 2k$

Substitute x_2 and x_3 in ①

$-2x_1 + 2k + 4k = 0$

$\Rightarrow 2x_1 = 6k$

$\Rightarrow x_1 = 3k$

i.e., $e_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3k \\ 2k \\ k \end{bmatrix} = k \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = k\, X_3$

$\therefore X_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$\therefore$ The eigen values of A are 3, 2, 5 and Correspondi

the eigen vectors are

$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

5. Find the characteristic roots of the matrix

$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

6. Find the Eigen values and Eigen vectors of the following matrices

2001, 2006 (a) $\begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$ 2005, 2012 (b) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ 2006, 2011 (c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

2006, 2008 (d) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ 2008, 2011 (e) $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ 2009 (f) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

2010 (g) $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ 2012 (h) $\begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -1 & 0 \end{bmatrix}$

Cayley - Hamilton Theorem

Statement: Every square matrix satisfies its own characteristic equation.

(or) If A is a square matrix of order n whose characteristic equation is

$$(-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \dots + k_n = 0 ; \quad \text{then}$$

$$(-1)^n A^n + k_1 A^{n-1} + k_2 A^{n-2} + \dots + k_n I = 0$$

Proof: Let A be an n square matrix

Let $D(\lambda)$ be characteristic polynomial of A, given by $D(\lambda) = |\lambda I - A|$

$$= \lambda^n + C_{n-1}\lambda^{n-1} + C_{n-2}\lambda^{n-2} + \dots + C_1 \lambda + C_0 \quad -①$$

Let $B(\lambda)$ be the adjoint of $(A - \lambda I)$

The elements of $B(\lambda)$ are cofactors of the matrix $(A - \lambda I)$ and are polynomials in λ of degree not exceeding $n-1$

Thus

$$B(\lambda) = B_{n-1}\lambda^{n-1} + B_{n-2}\lambda^{n-2} + \dots + B_1 \lambda + B_0 \quad -②$$

Where B_i are n square matrices whose elements are functions of the elements of A and independent of λ

We know that

$$(\lambda I - A) \operatorname{adj}(\lambda I - A) = (\lambda I - A) I$$

$$\Rightarrow (\lambda I - A) B(\lambda) = (\lambda I - A) I$$

from ① and ② we have

$$(\lambda I - A)(B_{n-1}\lambda^{n-1} + B_{n-2}\lambda^{n-2} + \cdots + B_1\lambda + B_0)$$
$$= I(\lambda^n + C_{n-1}\lambda^{n-1} + \cdots + C_1\lambda + C_0) \quad \text{--- (3)}$$

Equating the like powers of λ on both sides of (3)

we get $B_{n-1} = I$

$$B_{n-2} - AB_{n-1} = C_{n-1} I$$

$$B_{n-3} - AB_{n-2} = C_{n-2} I$$

$$\cdots\cdots$$

$$B_0 - AB_1 = C_1 I$$

$$-AB_0 = C_0 I$$

Multiplying both sides of the above matrices equations by $A^n, A^{n-1}, A^{n-2}, \ldots, A, I$ respectively we have

$$A^n B_{n-1} = A^n$$

$$A^{n-1}B_{n-2} - A^n B_{n-1} = A^{n-1}C_{n-1}$$

$$A^{n-2}B_{n-3} - A^{n-1}B_{n-2} = A^{n-2}C_{n-2}$$

$$\cdots\cdots$$

$$AB_0 - A^2B_1 = AC_1$$

$$-AB_0 = C_0 I$$

By Adding all the above equations, we get

$$0 = A^n + C_{n-1}A^{n-1} + C_{n-2}A^{n-2} + \cdots + C_1 A + C_0 I$$

Since all the terms on L.H.S cancel each other

Thus A satisfies its own characteristic equation

<u>Note</u>:- Proof is not required for this theorem.

Problems

1. Verify Cayley-Hamilton theorem for $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and hence find A^{-1} and A^8

Sol: Given $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}_{2\times 2}$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow |A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(-1-\lambda) - 4 = 0 \Rightarrow -1 - \lambda + \lambda + \lambda^2 - 4 = 0$$

$$\Rightarrow \lambda^2 - 5 = 0 \rightarrow (1)$$

Replace A in place of $\lambda \Rightarrow A^2 - 5I = 0$ — (2)

Now $A^2 = AA = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 5I$

i.e $A^2 - 5I = 5I - 5I = 0$

Thus 'A' satisfies the characteristic equation and hence Cayley-Hamilton theorem verified.

To find A^{-1}

Multiply $A^2 - 5I = 0$ by A^{-1}

$$\Rightarrow A^{-1}A^2 - 5IA^{-1} = 0 \Rightarrow A - 5A^{-1} = 0$$

$\Rightarrow$ [illegible] $\Rightarrow A = 5A^{-1}$

$$\Rightarrow A^{-1} = \frac{1}{5}A = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

To find A^8

Multiply eq (2) by A^6

$$\Rightarrow A^6(A^2 - 5I) = 0 \Rightarrow A^8 - 5A^6 = 0 \Rightarrow A^8 = 5A^6$$

$$\Rightarrow A^8 = 5(A^2 A^2 A^2) = 5 \cdot 5I \cdot 5I \cdot 5I = 625I$$

2. Verify Cayley Hamilton theorem for the following matrices and hence find A^{-1}

(a) $\begin{bmatrix} 2 & 5 \\ 1 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -3 \\ 7 & -4 \end{bmatrix}$

3. Verify Cayley-Hamilton theorem for $A = \begin{bmatrix} 1 & 4 & -3 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{bmatrix}$ and find A^{-1}

Sol: Given $A = \begin{bmatrix} 1 & 4 & -3 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 & -3 \\ 0 & 3-\lambda & 1 \\ 0 & 2 & -1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)[(3-\lambda)(-1-\lambda)-2] = 0$

$\Rightarrow (1-\lambda)[-3-3\lambda+\lambda+\lambda^2-2] = 0$

$\Rightarrow (1-\lambda)[\lambda^2-2\lambda-5] = 0$

$\Rightarrow \lambda^2-2\lambda-5-\lambda^3+2\lambda^2+5\lambda = 0$

$\Rightarrow -\lambda^3+3\lambda^2+3\lambda-5 = 0$

$\Rightarrow \lambda^3-3\lambda^2-3\lambda+5 = 0$ — (1)

Now replace matrix 'A' in place of scalar λ in (1)

i.e $A^3-3A^2-3A+5I = 0$ — (2)

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 4 & -3 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 4 & -3 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 4+12-6 & -3+4+3 \\ 0 & 9+2 & 3-1 \\ 0 & 6-2 & 0+2+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 10 & 4 \\ 0 & 11 & 2 \\ 0 & 4 & 3 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 1 & 10 & 4 \\ 0 & 11 & 2 \\ 0 & 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 & -3 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 4+30+8 & -3+10-4 \\ 0 & 33+4 & 11-2 \\ 0 & 12+6 & 4-3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 42 & 3 \\ 0 & 37 & 9 \\ 0 & 18 & 1 \end{bmatrix}$$

Now $A^3 - 3A^2 - 3A + 5I = A^3 + 5I - 3(A^2 + A)$

$$= \begin{bmatrix} 1 & 42 & 3 \\ 0 & 37 & 9 \\ 0 & 18 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} - 3\left\{\begin{bmatrix} 1 & 10 & 4 \\ 0 & 11 & 2 \\ 0 & 4 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 4 & -3 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 6 & 42 & 3 \\ 0 & 42 & 9 \\ 0 & 18 & 6 \end{bmatrix} - 3\begin{bmatrix} 2 & 14 & 1 \\ 0 & 14 & 3 \\ 0 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 42 & 3 \\ 0 & 42 & 9 \\ 0 & 18 & 6 \end{bmatrix} - \begin{bmatrix} 6 & 42 & 3 \\ 0 & 42 & 9 \\ 0 & 18 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore A^3 - 3A^2 - 3A + 5I = O$

Hence Cayley-Hamilton theorem verified.

<u>To find A^{-1}</u>

Multiply eq ② by A^{-1}

$\Rightarrow (A^3 - 3A^2 - 3A + 5I)A^{-1} = O \Rightarrow A^2 - 3A - 3I + 5A^{-1} = O$

$$\Rightarrow A^{-1} = \frac{1}{5}(3A + 3I - A^2) = \frac{1}{5}\left\{3\begin{bmatrix} 1 & 4 & -3 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 10 & 4 \\ 0 & 11 & 2 \\ 0 & 4 & 3 \end{bmatrix}\right\}$$

$$\Rightarrow A^{-1} = \frac{1}{5}\left\{\begin{bmatrix} 3 & 12 & -9 \\ 0 & 9 & 3 \\ 0 & 6 & -3 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 10 & 4 \\ 0 & 11 & 2 \\ 0 & 4 & 3 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\left\{\begin{bmatrix} 6 & 12 & -9 \\ 0 & 12 & 3 \\ 0 & 6 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 10 & 4 \\ 0 & 11 & 2 \\ 0 & 4 & 3 \end{bmatrix}\right\}$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 5 & 2 & -13 \\ 0 & 1 & 1 \\ 0 & 2 & -3 \end{bmatrix}$$

4. Verify Cayley - Hamilton theorem for $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

and find A^{-1}, $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$.

5. Find A^{-1} and A^4 by using Cayley-Hamilton theorem

if $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ Hint:- $|A - \lambda I| = \lambda^3 - 3\lambda^2 - \lambda + 9 = 0$

Diagonalization and powers of a matrix:

Similar Matrix:

A matrix 'A' is said to be similar to a matrix 'B' if there exists a non-singular matrix 'P' such that $B = P^{-1}AP$

This transformation of A to B is known as similarity transformation.

Note: 1 → Similar Matrices A and B have same eigen values

2 → Further if x is an eigen vector of A then $Y = P^{-1}x$ is an eigen vector of matrix B.

Proof: ① Suppose B is similar to A

i.e $B = P^{-1}AP$

Consider the characteristic polynomial to B

i.e $|B - \lambda I| = |P^{-1}AP - \lambda I|$

$= |P^{-1}AP - \lambda P^{-1}IP|$

$= |P^{-1}(A - \lambda I)P|$

$= |P^{-1}|\,|A - \lambda I|\,|P|$

$= |A - \lambda I|$ $\left[\text{Since } |P^{-1}||P| = |P||P^{-1}| = I\right]$

$\therefore |B - \lambda I| = |A - \lambda I|$

Thus A and B have same characteristic polynomial and therefore has the same eigen values proof ②: Let x be eigen vector of A, so that

$$Ax = \lambda x$$

Consider $B = P^{-1}AP$

post multiply by P^{-1}

$$\Rightarrow BP^{-1} = (P^{-1}AP)P^{-1} = P^{-1}A(PP^{-1}) = P^{-1}A$$

$$\Rightarrow BP^{-1} = P^{-1}A$$

Post multiply by X

$$\Rightarrow (BP^{-1})X = (P^{-1}A)X$$

$$\Rightarrow B(P^{-1}X) = P^{-1}(AX) = P^{-1}(\lambda X) = \lambda(P^{-1}X)$$

$$\therefore B(P^{-1}X) = \lambda(P^{-1}X)$$

Thus $P^{-1}X$ is an eigen vector of B corresponding to the eigen value λ.

<u>Diagonalization</u>: -

A n square matrix A with n-linearly independent eigen vectors is similar to a diagonal matrix D whose diagonal elements are the eigen values of n.

<u>proof</u>: Let A be an $n \times n$ square matrix

Let $x_1, x_2 \cdots x_n$ be the linearly independent eigen vectors of a matrix A corresponding to 'n' eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$

Construct P, known as model matrix, having $x_1, x_2, \cdots, x_n$ as the n column vector

i.e., $P = [x_1\ x_2 \cdots x_n]$

Similarly $D^3 = P^{-1}A^3P$

Thus $D^n = P^{-1}A^nP$

To obtain A^n, pre multiplying by P and post multiplying by P^{-1}

$\Rightarrow PD^nP^{-1} = PP^{-1}A^nPP^{-1}$

$= I A^n I$

$\Rightarrow A^n = PD^nP^{-1}$

<u>Problems</u>:-

1. Diagonalize the following matrices. Find the model matrix P which diagonalizes.

(a) $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

<u>Sol</u>: A is diagonalized by P whoose columns are the linearly independent eigen vectors of A

The characteristic equation of A is

$$|A-\lambda I| = \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0$$

$\Rightarrow (5-\lambda)^2 - 9 = 0 \Rightarrow 25 + \lambda^2 - 10\lambda - 9 = 0$

$\Rightarrow \lambda^2 - 10\lambda + 16 = 0$

$\Rightarrow \lambda^2 - 8\lambda - 2\lambda + 16 = 0$

$\Rightarrow (\lambda-8)(\lambda-2) = 0$

$\Rightarrow \lambda = 2, 8$

So $\lambda = 2, 8$ are two distinct eigen values

Since $x_1, x_2, \ldots, x_n$ are linearly independent and P^{-1} exists.

Consider

$$AP = A[x_1 \ x_2 \ \ldots \ x_n] = [Ax_1 \ Ax_2 \ \ldots \ Ax_n]$$

$$= [\lambda_1 x_1 \ \lambda_2 x_2 \ \ldots \ \lambda_n x_n]$$

$$= [x_1 \ x_2 \ \ldots \ x_n]\begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_3 & \ldots & 0 \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

where D is the diagonal matrix with eigen values of A as the principal diagonal elements

i.e., D is known as spectral matrix

Therefore $B = P^{-1}AP = P^{-1}(PD) = (P^{-1}P) \cdot D$

$$= D$$

$$\therefore \boxed{D = P^{-1}AP}$$

Powers of a matrix A:-

Consider $D = P^{-1}AP$

$$D^2 = (P^{-1}AP)(P^{-1}AP)$$

$$= P^{-1}A(PP^{-1})AP$$

$$= (P^{-1}A)I(AP) = P^{-1}AAP = P^{-1}A^2P$$

5. If $A = \begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}$ Then show that A is Hermitian and iA is Skew-Hermitian matrix (7

Sol: Given that $A = \begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}$.

$$\bar{A} = \begin{bmatrix} 2 & \overline{3+2i} & -4 \\ \overline{3-2i} & 5 & \overline{6i} \\ -4 & \overline{-6i} & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3-2i & -4 \\ 3+2i & 5 & -6i \\ -4 & 6i & 3 \end{bmatrix}$$

$$\bar{A}^T = \begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix} = A \quad \therefore \text{ A is Hermitian Matrix}$$

$$\text{Now } iA = i\begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix} = \begin{bmatrix} 2i & 3i-2 & -4i \\ 3i+2 & 5i & -6 \\ -4i & 6 & 3i \end{bmatrix}$$

$$\overline{iA} = \begin{bmatrix} \overline{2i} & \overline{3i-2} & \overline{-4i} \\ \overline{3i+2} & \overline{5i} & -6 \\ \overline{-4i} & 6 & \overline{3i} \end{bmatrix} = \begin{bmatrix} -2i & -3i-2 & 4i \\ -3i+2 & -5i & -6 \\ 4i & 6 & -3i \end{bmatrix}$$

$$\overline{iA}^T = \begin{bmatrix} -2i & -3i+2 & 4i \\ -3i-2 & -5i & 6 \\ 4i & -6 & -3i \end{bmatrix} = -\begin{bmatrix} 2i & 3i-2 & -4i \\ 3i+2 & 5i & -6 \\ -4i & 6 & 3i \end{bmatrix}$$

$$= -(iA)$$

iA is Skew-Hermitian Matrix

6: Show that $A = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$ is Hermitian. Find its eigen values and eigen vectors.

Solution: Given that $A = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 2 & 3-4i \\ 3+4i & 2 \end{bmatrix} \quad \text{Also} \quad \bar{A}^T = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix} = A$$

$\therefore$ A is Hermitian Matrix.

The Characterstic equation for A is

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & 3+4i \\ 3-4i & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)^2 - (3+4i)(3-4i) = 0$$

$$\Rightarrow 4 - 4\lambda + \lambda^2 - (9+16) = 0$$

$$\Rightarrow \lambda^2 - 4\lambda - 21 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 3\lambda - 21 = 0$$

$$\Rightarrow \lambda(\lambda-7) + 3(\lambda-7) = 0$$

$$\Rightarrow (\lambda-7)(\lambda+3) = 0$$

$$\Rightarrow \lambda = 7, -3$$

for $\lambda = -3$; $\Rightarrow (A-\lambda I)X = 0$
$\Rightarrow (A+3I)X = 0 \Rightarrow \begin{bmatrix} 5 & 3+4i \\ 3-4i & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow 5x_1 + (3+4i)x_2 = 0 \Rightarrow 5x_1 = -(3+4i)x_2$
$(3-4i)x_1 + 5x_2 = 0$
$\Rightarrow x_1 = -\frac{(3+4i)}{5} x_2$

If $x_1 = -(3+4i)$, then $x_2 = 5$

$$X_1 = \begin{bmatrix} -3-4i \\ 5 \end{bmatrix}$$

for $\lambda = 7 \Rightarrow (A-7I)X = 0 \Rightarrow \begin{bmatrix} -5 & 3+4i \\ 3-4i & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-5x_1 + (3+4i)x_2 = 0 \Rightarrow 5x_1 = (3+4i)x_2$ If $x_1 = 3+4i$, $x_2 = 5$
$(3-4i)x_1 - 5x_2 = 0$

$$\Rightarrow X_2 = \begin{bmatrix} 3+4i \\ 5 \end{bmatrix}$$

(8)

7. Show that $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ is skew-Hermitian matrix and also unitary. Find the eigen values & eigen vectors.

Sol: Given $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ and $\bar{A} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} \Rightarrow$

$$\bar{A}^T = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} = -\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} = -A$$

$\therefore$ A is skew-Hermitian matrix

Now $$A\bar{A}^T = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}\begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} = \begin{bmatrix} -i^2 & 0 & 0 \\ 0 & -i^2 & 0 \\ 0 & 0 & -i^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is unitary matrix

Now the characteristic equation of A is

$$|A - \lambda I| = \begin{vmatrix} i-\lambda & 0 & 0 \\ 0 & 0-\lambda & i \\ 0 & i & 0-\lambda \end{vmatrix} = 0 \Rightarrow (i-\lambda)(\lambda^2+1) = 0$$
$$\Rightarrow \lambda^3 - i\lambda^2 + \lambda - i = 0$$
$$\Rightarrow (\lambda+i)(\lambda-i)^2 = 0$$
$$\Rightarrow \lambda = -i, i, i$$

$\therefore$ The eigen values of A are $\lambda = -i, i, i$ which are purely imaginary (for skew-Hermitian) and are of absolute values unity [i.e. $|-i| = |i| = 1$]

for $\lambda = -i \Rightarrow (A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 2i & 0 & 0 \\ 0 & i & i \\ 0 & i & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 2ix_1 = 0 \Rightarrow x_1 = 0 \\ ix_2 + ix_3 = 0 \Rightarrow x_2 = -x_3 \end{matrix}$$

$$\therefore X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

for $\lambda = i \Rightarrow (A - \lambda I)X = 0 \Rightarrow (A - iI)X = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -i & i \\ 0 & i & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} -ix_2 + ix_3 = 0 \\ ix_2 - ix_3 = 0 \end{matrix}\right\} \Rightarrow x_2 = x_3$$

if $x_1 = k$ then $x_2 = x_3 \Rightarrow x_3 = l$

$$\Rightarrow X_3 = \begin{bmatrix} k \\ l \\ l \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + l\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ \& } X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Problems:

1. Prove that $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ is Hermitian matrix & find its eigen Val

2. P.T $A = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$ is Skew-Hermitian & find its eigen values

3. P.T $C = \begin{bmatrix} \frac{i}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{i}{2} \end{bmatrix}$ is unitary matrix. Find its eigen values.

4. If $A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$ show that $(I-A)(I+A)^{-1}$ is a unitary matrix.

5. Find the eigen vectors of the Hermitian matrix $A = \begin{pmatrix} a & b+ic \\ b-ic & k \end{pmatrix}$

6. Show that the column vectors of matrix unitary matrix $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ from an orthogonal System

7. Show that $A = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$ is unitary if $a^2+b^2+c^2+d^2 = 1$

8. P.T every Hermitian matrix can be written as $A+iB$ whose A is real Symmetric & B is real skew-Symmetric.

Eigen vector corresponding $\lambda = 2$

We have $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 5-2 & 3 \\ 3 & 5-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{array}{l} 3x_1 + 3x_2 = 0 \\ 3x_1 + 3x_2 = 0 \end{array}\right\} - (1)$$

$$\Rightarrow 3x_1 = -3x_2 \Rightarrow x_1 = -x_2$$

If $x_2 = -k$, then $x_1 = k$

$$\text{i.e., } e_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ -k \end{bmatrix} = k\begin{bmatrix} 1 \\ -1 \end{bmatrix} = kX_1$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Eigen vector corresponding $\lambda = 8$

We have $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 5-8 & 3 \\ 3 & 5-8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} -3x_1 + 3x_2 = 0 \\ +3x_1 - 3x_2 = 0 \end{array} \Rightarrow 3x_1 = 3x_2 \Rightarrow x_1 = x_2$$

If $x_1 = k$ then $x_2 = k$

$$\text{i.e., } e_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \end{bmatrix} = kX_2 \quad \therefore X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Now model matrix $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

To find P^{-1}:-

$$P^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{-1-1}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Diagonalization:- $D = P^{-1}AP$

$$\Rightarrow D = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}5 & 3\\3 & 5\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}8 & 8\\2 & -2\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}16 & 0\\0 & 4\end{bmatrix} = \begin{bmatrix}8 & 0\\0 & 2\end{bmatrix}$$

which is the required diagonal matrix

(b) Find A^8 for $A = \begin{bmatrix}1 & 1 & 1\\0 & 2 & 1\\-4 & 4 & 3\end{bmatrix}$

Sol: The characteristic equation of A is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix}1-\lambda & 1 & 1\\0 & 2-\lambda & 1\\-4 & 4 & 3-\lambda\end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda)-4]-0+(-4)[1-(2-\lambda)]=0$$

$$\Rightarrow (1-\lambda)[6-5\lambda+\lambda^2-4]-4(\lambda-1)=0$$

$$\Rightarrow 6-5\lambda+\lambda^2-4-6\lambda+5\lambda^2-\lambda^3+4\lambda-4\lambda+4=0$$

$$\Rightarrow -\lambda^3+6\lambda^2-11\lambda+6=0$$

$$\Rightarrow \lambda^3-6\lambda^2+11\lambda-6=0$$

1	1	−6	11	−6
	0	1	−5	6
	1	−5	6	(0)

$$\Rightarrow (\lambda-1)(\lambda^2-5\lambda+6)=0$$

$$\Rightarrow (\lambda-1)(\lambda-2)(\lambda-3)=0$$

$$\Rightarrow \lambda = 1, 2, 3$$

Eigen values of A are $\lambda_1=1,\ \lambda_2=2,\ \lambda_3=3$

— — — — .

Eigen Vector corresponding $\lambda = 1$

We have $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (A - I)X = 0 \Rightarrow \begin{bmatrix} 1-1 & 1 & 1 \\ 0 & 2-1 & 1 \\ -4 & 4 & 3-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ -4 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_2 + x_3 = 0 - (1) \\ x_2 + x_3 = 0 - (2) \\ -4x_1 + 4x_2 + 2x_3 = 0 - (3) \end{matrix}$$

from (1) & (2) $\Rightarrow x_2 = -x_3$ if $x_3 = k$ then $x_2 = -k$

Substitute x_2 and x_3 in (3) $\Rightarrow -4x_1 - 4k + 2k = 0$

$\Rightarrow -4x_1 - 2k = 0 \Rightarrow 4x_1 = -2k$

$\Rightarrow x_1 = -\frac{k}{2}$

$$\text{i.e., } e_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k/2 \\ -k \\ k \end{bmatrix} = \frac{k}{2} \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} = \frac{k}{2} X_1$$

$$\therefore X_1 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$

Eigen Vector corresponding $\lambda = 2$

We have $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 1-2 & 1 & 1 \\ 0 & 2-2 & 1 \\ -4 & 4 & 3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} -x_1 + x_2 + x_3 = 0 - (1) \\ x_3 = 0 - (2) \\ -4x_1 + 4x_2 + x_3 = 0 - (3) \end{matrix}$$

Substitute x_3 in (1)

$\Rightarrow -x_1 + x_2 = 0 \Rightarrow x_1 = x_2$

if $x_2 = k$ then $x_1 = k$

$$\text{i.e., } e_2 = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = k\, X_2$$

$$\therefore X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Eigen Vector Corresponding $\lambda = 3$,

we have $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 1-3 & 1 & 1 \\ 0 & 2-3 & 1 \\ -4 & 4 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ -4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} -2x_1 + x_2 + x_3 &= 0 \\ -x_2 + x_3 &= 0 \Rightarrow x_2 = x_3 \\ -4x_1 + 4x_2 &= 0 \Rightarrow x_1 = x_2 \end{aligned} \quad \Rightarrow x_1 = x_2 = x_3 = k \text{ say}$$

$$\text{i.e., } e_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = k \cdot X_3$$

$$\therefore X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Now the model matrix $P = [X_1 \; X_2 \; X_3]$

$$= \begin{bmatrix} -1 & 1 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

To find P^{-1} :- By Gauss Jordan Method

$$A|I = \left[\begin{array}{ccc|ccc} -1 & 1 & 1 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{R_{31}(2)}{R_{21}(-2)} \sim \begin{bmatrix} -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 2 & 3 & 2 & 0 & 1 \end{bmatrix} \underset{R_{32}(2)}{\sim} \begin{bmatrix} -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{bmatrix}$$

$$\underset{R_{23}(1)}{R_{13}(-1)} \sim \begin{bmatrix} -1 & 1 & 0 & 3 & -2 & -1 \\ 0 & -1 & 0 & -4 & 3 & 1 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{bmatrix}$$

$$R_{12}(1) \sim \begin{bmatrix} -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -4 & 3 & 1 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{bmatrix} \underset{R_{2}(-1)}{\overset{R_1(-1)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 4 & -3 & -1 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{bmatrix}$$

$$= I/P^{-1}$$

$$\therefore P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$$

Now Diagonal matrix $D = P^{-1}AP$

$$\Rightarrow D = \begin{bmatrix} 1 & -1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 8 & -6 & -2 \\ -6 & 6 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

which is the required diagonal matrix

$$A^8 = PD^8P^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ -2 & 1 & 1 \\ +2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 6561 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 256 & 6561 \\ -2 & 256 & 6561 \\ +2 & 0 & 6561 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+1024-13122 & 1-768+13122 & -256+6561 \\ -2+1024-13122 & 2-768+13122 & -256+6561 \\ 2+0+13122 & -2+0+13122 & 0+0+6561 \end{bmatrix}$$

$$= \begin{bmatrix} -12099 & 12355 & 6305 \\ -12100 & 12356 & 6305 \\ -13120 & 13120 & 6561 \end{bmatrix}$$

(c) Diagonalize the matrix $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$, $\lambda = 1, \pm\sqrt{5}$

(d) Diagonalize the matrix $A = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix}$

(e) Diagonalize the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

(f) (i) Diagonalize $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$ and (ii) Diagonalize $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

and find A^4

UNIT III :- Linear Transformation.

Topics:
1. Real and ~~Sym~~ Complex Matrices.
2. Properties of Real and Complex Matrices.
3. Quadratic forms.
4. Reduction of Quadratic form to Canonical form
5. Sylvester law and Singular Value decomposition.

Real and Complex Matrices:

Real Matrix:

A matrix $A = (a_{ij})$ is said to be real matrix if every element a_{ij} of A is real.

Symmetric Matrix:-

A real square matrix is said to be ~~Skew~~ symmetric matrix, if $A = A^T$ are $a_{ij} = a_{ji}$

Skew - Symmetric Matrix:-

A real square matrix is said to be Skew-symmetric if $A^T = -A$ i.e $a_{ji} = -a_{ij}$

Orthogonal Matrix:-

If $A^T = A^{-1}$ then the real square matrix is called the orthogonal matrix

Properties:

(1) $a_{ii} = 0$ for a skew symmetric matrix

Since $A^T = -A \Rightarrow a_{ji} = -a_{ij}$

$\Rightarrow a_{ii} = -a_{ii}$ for $i = j$

$\Rightarrow a_{ii} + a_{ii} = 0 \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$

(II) A is symmetric $\Rightarrow A^T$ is symmetric

(III) A is symmetric $\Rightarrow A^{-1}$ is also symmetric

(IV) A and B are symmetric $\Rightarrow A+B$ and $A-B$ are also symmetric

proof: Since A and B are symmetric

$$A = A^T \text{ and } B = B^T$$

Now $(A+B)^T = A^T + B^T$

$= A+B$

& $(A-B)^T = A^T - B^T$

$= A-B$

V) If A and B are symmetric then their product AB is symmetric $\Longleftrightarrow AB = BA$

proof: Given A and B are symmetric

$$A = A^T \text{ and } B = B^T$$

Suppose AB is symmetric $\Longleftrightarrow (AB)^T = AB$

$\Longleftrightarrow B^T A^T = AB$

$\Longleftrightarrow BA = AB$

I) Every square matrix can be written as the sum of symmetric matrix and skew-symmetric matrix

proof: Let A be a square matrix

Consider $B = \frac{1}{2}(A + A^T)$

$B^T = \frac{1}{2}(A+A^T)^T = \frac{1}{2}[A^T + (A^T)^T]$

$= \frac{1}{2}(A^T + A) = B$

So, B is symmetric

Similarly $C = \frac{1}{2}(A - A^T)$

$C^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - (A^T)^T)$

$= \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -C$

So, C is Skew-Symmetric matrix

Now $A = B+C = \left[\frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)\right]$

$= \frac{1}{2}\left[A+A^T+A-A^T\right] = \frac{2A}{2} = A$

$\therefore A = B+C.$

Hence every Square matrix can be written as Sum of Symmetric matrix and Skew-Symmetric matrix

(VII) The determinent of orthogonal matrix ±1

proof: i.e $A^T = A^{-1}$

$\Rightarrow A^T A = I \Rightarrow |A^T A| = |I| \Rightarrow |A||A^T| = 1$

$\Rightarrow |A||A| = 1 \Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1$

(VIII) Product of Two orthogonal matrix is orthogonal.

proof: Let A and B be two orthogonal matrices.

i.e, $A^T = A^{-1}$ and $B^T = B^{-1}$

Now $(AB)(AB)^T = (AB)(B^T A^T)$

$= A(BB^T)A^T$

$= AIA^T = AA^T = I$

$\Rightarrow (AB)^T = (AB)^{-1}$

So, product of two orthogonal matrices is orthogo

Orthogonal Transformation:

Which geometrically represents a rotation, is a transformation $Y = AX$.

where A is an orthogonal matrix

Norm of a vector x denoted by $\|x\|$

$\|x\| = \sqrt{x^T x}$ represents the length of the vector

A transformation $X = AY$ which transforms a vector X to another vector Y through an orthogonal matrix 'A' is called an orthogonal transformation.

Complex Matrix:

A matrix whose elements are complex (or) real number's is called a Complex Matrix

Conjugate of A Matrix:-

A matrix obtained from another matrix 'A' by replacing the elements of 'A' with their complex conjugate is called conjugate matrix of A.

$$A = \begin{bmatrix} 2+3i & 5 \\ 6-7i & 5+i \end{bmatrix} \quad \bar{A} = \begin{bmatrix} 2-3i & 5 \\ 6+7i & 5-i \end{bmatrix}$$

Conjugate Transpose of A:-

The transpose of conjugate matrix of a matrix is called conjugate transpose of that matrix.

In general the conjugate transpose of a matrix 'A' is denoted by $\bar{A}^T$ or A^θ

Hermitian Matrix:-

A square matrix A is said to be Hermitian if $A^\theta = A$ i.e $\bar{A}^T = A$

Eg:- $$A = \begin{bmatrix} 1 & 2-i & 3+i \\ 2+i & 2 & -7i \\ 3-i & 7i & 5 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 1 & 2+i & 3-i \\ 2-i & 2 & 7i \\ 3+i & -7i & 5 \end{bmatrix}$$

$$\bar{A}^T = \begin{bmatrix} 1 & 2-i & 3+i \\ 2+i & 2 & -7i \\ 3-i & 7i & 5 \end{bmatrix} = A$$

Skew-Hermitian Matrix:

A square matrix 'A' is said to be Skew-Hermitian Matrix if $\bar{A}^T = A^\Theta = -A$

Eg: $\begin{bmatrix} 2i & 3-i & 4+3i \\ -3-i & -4i & 6+5i \\ 3i-4 & 5i-6 & 0 \end{bmatrix}$

Unitary Matrix

A square matrix 'A' is said to be Unitary if $A^\Theta = A^{-1}$ or $\bar{A}^T = A^{-1}$ or $A\bar{A}^T = \bar{A}^T A = I$

Properties:

1. The principal diagonal elements of a Hermitian matrix are real.
(or) The eigen values of a Hermitian matrix are real.

proof: Let A be Hermitian matrix

i.e $A^\Theta = A$ — ①

If x be the eigen vector corresponding to the eigen value λ of A, then $Ax = \lambda x$ → ②

premultiplying both sides of eq ② by $\bar{x}^T$, we get

$$\bar{x}^T A x = \bar{x}^T \lambda x \Rightarrow \bar{x}^T A x = \lambda \bar{x}^T x$$

$$\Rightarrow x^\Theta A x = \lambda x^\Theta x \quad — ③$$

Taking the conjugate Transpose on both sides.

$$\overline{(\bar{x}^T A x)}^T = \overline{(\lambda \bar{x}^T x)}^T$$

$\Rightarrow \bar{A} = (\bar{A}^{T})^{T} \Rightarrow \bar{A} = (A^{T})^{-1}$

$\Rightarrow (\overline{A^{T}})^{T} = (A^{T})^{-1} \Rightarrow (\overline{A^{T}})^{T} = (A^{T})^{-1}$

i.e A^{T} is unitary matrix

9. The eigen values of an unitary matrix are having an absolute ~~eigen vector x.~~ of 1.

proof: let λ be an eigen value of an unitary matrix 'A' whose corresponding eigen vector x.

$\therefore A^{\theta} = \bar{A}^{T} = A^{-1} \rightarrow (1)$

and $AX = \lambda X \quad -(2)$

$(AX)^{\theta} = (\lambda X)^{\theta} \quad -(3)$

(3)×(2) $\quad (AX)^{\theta}(AX) = (\lambda X)^{\theta}(\lambda X)$

$\Rightarrow (X^{\theta}A^{\theta})(AX) = \bar{\lambda} X^{\theta} \cdot \lambda X$

$\Rightarrow X^{\theta} \cdot (A^{\theta}A) X = \bar{\lambda}\lambda X^{\theta} X$

$\Rightarrow X^{\theta} I X = \bar{\lambda}\lambda X^{\theta} X$

$\Rightarrow X^{\theta}X = \bar{\lambda}\lambda X^{\theta}X \Rightarrow \bar{\lambda}\lambda = 1$

$\Rightarrow |\lambda|^{2} = 1 \Rightarrow |\lambda| = 1$

$\therefore$ The eigen value of an unitary matrix are having an absolute value of 1.

Problems: Express the matrix A as a sum of symmetric and skew-symmetric matrix where $A = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$

and $A^T = \begin{bmatrix} 3 & 2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix}$

Consider $B = \frac{(A + A^T)}{2} = \frac{1}{2}\left\{\begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix}\right\}$

$= \frac{1}{2}\begin{bmatrix} 6 & 0 & 11 \\ 0 & 14 & 3 \\ 11 & 3 & 0 \end{bmatrix}$ is Symmetric Matrix

and $C = \frac{1}{2}(A - A^T) = \frac{1}{2}\left\{\begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix}\right\}$

$= \frac{1}{2}\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix}$ is Skew Symmetric Matrix

Now $B + C = \frac{1}{2}\begin{bmatrix} 6 & 0 & 11 \\ 0 & 14 & 3 \\ 11 & 3 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 6 & -4 & 12 \\ 4 & 14 & -2 \\ 10 & 8 & 0 \end{bmatrix}$

$= \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix} = A$

Hence $A = B + C$ which is sum of a symmetric and skew-symmetric matrix.

2. Determine the values of a, b, c when $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal

Sol: We know that if A is orthogonal then $A^T = A^{-1}$

or $A^TA = AA^T = I$

so that $AA^T = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix} \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix}$

$$= \begin{bmatrix} 4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$-2b^2+c^2 = 0 \Rightarrow 2b^2 = c^2$
$\Rightarrow c = \pm\sqrt{2}b$

$4b^2+c^2=1$ $\quad$ $2b^2-c^2=0$

$2b^2-c^2=0$ $\quad$ $a^2+b^2+c^2=1$ $\quad$ $a^2-b^2-c^2=0$

$-2b^2+c^2=0$ $\quad$ $a^2-b^2-c^2=0$ $\quad$ $a^2+b^2+c^2=1$

from $a^2-b^2-c^2=0 \Rightarrow a^2 = b^2+c^2 = b^2+2b^2 = 3b^2$
$\Rightarrow a = \pm\sqrt{3}b$

from $4b^2+c^2=1 \Rightarrow 4b^2+2b^2=1 \Rightarrow 6b^2=1 \Rightarrow b^2 = \frac{1}{6} \Rightarrow b = \pm 1/\sqrt{6}$

and $a = \pm\frac{1}{\sqrt{2}}$ and $c = \pm\frac{1}{\sqrt{3}}$

$\therefore a = \pm\frac{1}{2}, b = \pm\frac{1}{\sqrt{3}}$ & $c = \pm\frac{1}{\sqrt{3}}$ if A is orthogonal

3. Is the matrix $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$ is orthogonal?

4. Show that $A = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$ is orthogonal

Linear Transformation:

A transformation $x = Ay$ which transforms a vector 'x' to another vector 'y' through a matrix 'A' is called a linear transformation.

Orthogonal Transformation:

A transformation $x = Ay$ which transforms a vector 'x' to another vector 'y' through an orthogonal matrix 'A' is called an orthogonal matrix.

Orthogonal vectors:

Two vectors 'x_1' and 'x_2' are said to be orthogonal if $x_1^T x_2 = 0$

Orthogonal set of vectors:

A set of vectors is said to be an orthogonal set of vectors if every pair of two distinct vectors in that set are orthogonal i.e A set of vectors $x_1, x_2, \ldots, x_n$ are said to be orthogonal set of vectors if $x_i^T x_j = 0$ for $i \neq j$

Orthogonal set of vectors:

A set of vectors $x_1, x_2, \ldots x_n$ are said to be an orthogonal set of vectors if $x_i^T x_j = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$

Quadratic forms:

A homogeneous equation in 'n'-variables of degree 2, is called a quadratic form which is denoted by Q.

A quadratic form in 'n' – variables will be in the form $Q = X^T A X = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$

$$= a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + \cdots + a_{1n}x_1x_n$$
$$+ a_{21}x_1x_2 + a_{22}x_2^2 + a_{23}x_2x_3 + \cdots + a_{2n}x_2x_n$$
$$+ \cdots \cdots + \cdots + \cdots$$
$$+ a_{n1}x_nx_1 + a_{n2}x_nx_2 + \cdots + a_{nn}x_n^2$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$ & $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$

The matrix 'A' is called the coefficient matrix of the quadratic form Q.

Note: In general we assume that the coeff matrix 'A' of a quadratic form $Q = X^T A X$ is a symmetric Matrix.

Rank of A Quadratic form

The rank of the coefficient matrix of a quadratic form is called the rank of that quadratic form.

The Quadratic form $Q = X^T A X$ is said to be Singular if Rank of the quadratic form is less than no of variables [$|A| = 0$, i.e., if A is Singular matrix]

A quadratic form $Q = X^T A X$ is said to be non-singular if Rank of that quadratic form is equal to no of variables [i.e if A is non-singular $|A| \neq 0$]

Canonical Form of Sum of the Squares form (or) Normal Form of a Quadratic Form

A cannonical form of a real quadratic form $Q = X^T A X$ is $Y^T D Y$ or $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$ — ①

which can be obtained by an orthogonal transformation $X = PY$ where P is called the Modal matrix and D is the spectral matrix.

Here D is a diagonal matrix whose diagonal entries are nothing but the eigen values of A

Index:- The no of +ve square terms in the cannonical form of a quadratic form is defined as index of the quadratic form.

(or) The no. of positive eigen values of the coefficients matrix of a quadratic form is defined as index of the quadratic form.

Signature:- The difference between the number of positive and negative squares terms in the cannonical form of a quadratic form is called Signature of that quadratic form.

(or) The difference between the no of positive eigen values and no of negative eigen values of the coefficient matrix of a quadratic form is called the signature of the quadratic form.

Nature (or) definiteness of a Quadratic form.

Let r, n and s represent the rank, no of variables and index respectively of a quadratic form, then

(i) A quadratic form is said to be positive definite if $r=n$ and $s=n$ (or) if all the eigen values of its coefficient matrix are positive (+ve)

(ii) A quadratic form is said to be negative definite if $r=n$ and $s=0$ (or) if all the eigen values of its coefficient matrix are negative (-ve)

(iii) A quadratic form is said to be +ve semi definite if $r<n$ and $s=r$ (or) the eigen values of its coefficient matrix are such that atleast one eigen value is zero and remaining all are positive.

(iv) A quadratic form is said to be -ve semi definite if $r<n$ and $s=0$ (or) the eigen values of its coefficient matrix are such that atleast one eigen value is zero and remaining all are negative

(v) A quadratic form is said to be indefinite if ($s \neq r$ and $s \neq 0$) its index is neither equal to its rank nor equal to zero (or) its coefficient matrix has both positive and negative.

Problems: If $X_1 = \frac{1}{3}[2 \ -1 \ 2]^T$ & $X_2 = k[3 \ -4 \ -5]^T$ where $k = \frac{1}{\sqrt{50}}$, construct an orthogonal matrix $A = [X_1 \ X_2 \ X_3]$

Solution:- Let $X_3 = [a_1 \ a_2 \ a_3]^T$ be the undetermined vector

Since A is an orthogonal, vectors the column vectors of A form an orthogonal set of vectors. $X_i^T X_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

Now $X_1^T X_2 = \begin{bmatrix} \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 3k \\ -4k \\ -5k \end{bmatrix} = 2k + \frac{4}{3}k - \frac{10}{3}k = \frac{10k}{3} - \frac{10k}{3} = 0$

$\therefore X_1$ & X_2 are orthogonal set of vectors.

$X_1^T X_3 = \begin{bmatrix} \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{2}{3}a_1 - \frac{1}{3}a_2 + \frac{2}{3}a_3 = 0$ — ①

& $X_2^T X_3 = [3k \ -4k \ -5k] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 3ka_1 - 4ka_2 - 5ka_3 = 0$ — ②

& $X_3^T X_3 = [a_1 \ a_2 \ a_3] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_1^2 + a_2^2 + a_3^2 = 1$

$\Rightarrow 1 = \|X\| = \sqrt{a_1^2 + a_2^2 + a_3^2} \Rightarrow a_1^2 + a_2^2 + a_3^2 = 1$ — ③

from ① $2a_1 - a_2 + 2a_3 = 0$ — ④

$3ka_1 - 4ka_2 - 5ka_3 = 0 \Rightarrow 3a_1 - 4a_2 - 5a_3 = 0$ — ⑤

$a_1^2 + a_2^2 + a_3^2 = 1$ — ⑥

$4 \times ④ - ⑤ \Rightarrow 8a_1 - 4a_2 + 8a_3 = 0$

$3a_1 - 4a_2 - 5a_3 = 0$

$5a_1 + 13a_3 = 0 \Rightarrow a_1 = -\frac{13}{5}a_3$

Sub a_1 in ④ $\Rightarrow 2\left(-\frac{13}{5}a_3\right) - a_2 + 2a_3 = 0$

$\Rightarrow a_2 = -\frac{16}{5}a_3$

Sub a_1 & a_2 in ⑥ i.e. $a_1^2 + a_2^2 + a_3^2 = 1$

$\Rightarrow \left(-\frac{13}{5}a_3\right)^2 + \left(-\frac{16}{5}a_3\right)^2 + a_3^2 = 1$

$\Rightarrow \left(\frac{169}{25} + \frac{256}{25} + 1\right)a_3^2 = 1 \Rightarrow \frac{450}{25}a_3^2 = 1 \Rightarrow a_3^2 = \frac{25}{450} = \frac{1}{18}$

$\Rightarrow a_3 = \frac{1}{\sqrt{18}} = k_1$

Hence $a_1 = -\frac{13}{5}k_1$, $a_2 = -\frac{16}{5}k_1$ & $a_3 = k_1$

Thus the required orthogonal matrix A is

$$A = \begin{bmatrix} 2/3 & 3k & -13/5k_1 \\ -1/3 & -4k & -16/5k_1 \\ 2/3 & -5k & k_1 \end{bmatrix}$$

2. Find the real Symmetric matrix C of the Quadratic form $Q = x_1^2 + 3x_2^2 + 2x_3^2 + 2x_1x_2 + 6x_2x_3$

Solution: The Coefficient matrix A of Quadratic form

$$Q = x^T A x \text{ is}$$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}, \quad \text{so } C = \text{Symmetric Matrix} = \frac{1}{2}(A + A^T)$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 6 & 2 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 2 & 2 & 0 \\ 2 & 6 & 6 \\ 0 & 6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 3 \\ 0 & 3 & 2 \end{bmatrix}$$

i.e. $Q = x^T A x = x_1^2 + 3x_2^2 + 2x_3^2 + 2x_1x_2 + 6x_2x_3$

$$= x_1^2 + 3x_2^2 + 2x_3^2 + x_1x_2 + x_2x_1 + 3x_2x_3 + 3x_3x_2$$

$$= x_1^2 + x_1x_2 + 0\cdot x_1x_3 + 3x_2^2 + x_2x_1 + 3x_2x_3 + 2x_3^2 + 0\cdot x_3x_1 + 3x_3x_2$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 3 \\ 0 & 3 & 2 \end{bmatrix}$$

3. Write the quadratic form corresponding to the matrix

$$A = \begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$$

Sol: Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $A = \begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$

The quadratic form corresponding to the real symmetric matrix

$$Q = X^T A X = [x \;\; y \;\; z]\begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [x \;\; y \;\; z]\begin{bmatrix} 5y - z \\ 5x + y + 6z \\ -x + 6y + 2z \end{bmatrix} = x(5y - z) + y(5x + y + 6z) + z(-x + 6y + 2z)$$

$$= 5xy - xz + 5xy + y^2 + 6yz - zx + 6yz + 2z^2$$

$$= y^2 + 2z^2 + 10xy + 12yz - 2zx$$

4. Determine the nature, index and signature of the Quadratic form $2x_1^2+2x_2^2+3x_3^2+2x_1x_2-4x_1x_3-4x_2x_3$.

Solution: The real symmetric matrix A associated with the

Quadratic form $Q = X^TAX = 2x_1^2 + x_1x_2 - 2x_1x_3 + 2x_2^2 + x_2x_1 - 2x_2x_3 - 2x_3x_1 - 2x_3x_2 + 3x_3^2$

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}$$

The characteristic equation is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & -2 \\ 1 & 2-\lambda & -2 \\ -2 & -2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)[(2-\lambda)(3-\lambda)-4]-1[(3-\lambda)-4] -2[-2+2(2-\lambda)]=0$$

$$\Rightarrow (2-\lambda)[6-5\lambda+\lambda^2-4]-[3-\lambda-4]-2[-2+4-2\lambda]=0$$

$$\Rightarrow (2-\lambda)[\lambda^2-5\lambda+2]-(-1-\lambda)-2(-2\lambda+2)=0$$

$$\Rightarrow 2\lambda^2-10\lambda+4-\lambda^3+5\lambda^2-2\lambda+1+\lambda+4\lambda-4=0$$

$$\Rightarrow -\lambda^3+7\lambda^2-7\lambda+1=0 \Rightarrow \lambda^3-7\lambda^2+7\lambda-1=0$$

$$\Rightarrow (\lambda-1)(\lambda^2-6\lambda+1)=0$$

1	1	-7	7	-1
	0	1	-6	1
	1	-6	1	0

$$\Rightarrow \lambda-1=0 \text{ or } \lambda^2-6\lambda+1=0 \Rightarrow \lambda = \frac{6\pm\sqrt{36-4}}{2} = \frac{6\pm\sqrt{32}}{2} = \frac{6\pm 4\sqrt{2}}{2} = 3\pm2\sqrt{2}$$

$$= 3+2\sqrt{2},\ 3-2\sqrt{2}$$

$\Rightarrow \lambda = 1, 0.17, 5.828$, which are all +ve

Nature of the Quadratic form Q is positive definite.

Index = No of +ve eigen values = 3

Signature = diff between the +ve and -ve eigen values

= 3-0 = 3.

Reduction of Quadratic form to canonical form

Any quadratic form may be reduced to canonical by means of the following methods

1. Diagonalisation [Reduction to canonical form to canonical using Linear Transformation or Linear transformation of Quadratic form]

2. Orthogonalisation [Reduction to canonical form using Orthogonal transformation]

3. Laglange's Reduction

Reduction to canonical form using Linear Transformation

i.e Diagonalization of Symmetric Matrix

Procedure:

Step1: Write the Symmetric Matrix 'A' of the quadratic form

Step2: Write the matrix A in the following relation: $A_{n\times n} = I_n A I_n$

Step3: Reduce the matrix A on left hand side to a diagonal matrix (i) by applying elementary row operations on the left identity matrix and on A on left side (ii) By applying elementary column operations on the right identity matrix and on A on left hand side

Step4: By these operations $A = IAI$ will be reduced to the form $D = P^T A P$

where D is the diagonal matrix with elements $d_1, d_2, \ldots d_n$ and P is the matrix used in the linear transfor.

The canonical form is given by $Y^T D Y = [y_1\ y_2 \ldots y_n] \begin{bmatrix} d_1 & 0 & \ldots & 0 \\ 0 & d_2 & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & d_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

$$= d_1 y_1^2 + d_2 y_2^2 + \ldots + d_n y_n^2$$

Here Some of the coefficients $d_1, d_2, \ldots, d_n$

1. Find the nature of the quadratic form, index & signature of $10x^2+2y^2+5z^2-4xy-10xz+6yz$

Sol: The given quadratic form is $10x^2+2y^2+5z^2-4xy-10xz+6yz$ — ①

The Coefficient matrix of the Quadratic form is

$$A = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}_{3\times3}$$

Consider $A_{3\times3} = I_{3\times3} A_{3\times3} I_{3\times3}$

$$\begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow R_2 \to 5R_2 + R_1$, $R_3 \to 2R_3 + R_1$ $\sim$

$$\begin{bmatrix} 10 & -2 & -5 \\ 0 & 8 & 10 \\ 0 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to 2R_3 - R_2$ $\sim$

$$\begin{bmatrix} 10 & -2 & -5 \\ 0 & 8 & 10 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & -5 & 4 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to 5C_2 + C_1$, $C_3 \to 2C_3 + C_1$ $\sim$

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 20 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & -5 & 4 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$C_3 \to 2C_3 - C_2$

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & -5 & 4 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & -5 \\ 0 & 0 & 4 \end{bmatrix}$$

$R_1 \to \frac{R_1}{\sqrt{10}}, R_2 \to \frac{R_2}{\sqrt{40}}$

$$\begin{bmatrix} \frac{10}{\sqrt{10}} & 0 & 0 \\ 0 & \frac{40}{\sqrt{40}} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{10}} & 0 & 0 \\ \frac{1}{\sqrt{40}} & \frac{5}{\sqrt{40}} & 0 \\ 1 & -5 & 4 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & -5 \\ 0 & 0 & 4 \end{bmatrix}$$

$C_1 \to \frac{C_1}{\sqrt{10}}, C_2 \to \frac{C_2}{\sqrt{40}}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{10}} & 0 & 0 \\ \frac{1}{\sqrt{40}} & \frac{5}{\sqrt{40}} & 0 \\ -1 & -5 & 4 \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{40}} & 1 \\ 0 & \frac{5}{\sqrt{40}} & -5 \\ 0 & 0 & 4 \end{bmatrix}$$

~~Hence~~ Thus the normal form of the Quadratic form

$CF = y_1^2 + y_2^2 = P'DY$

where $P = \begin{bmatrix} \frac{1}{\sqrt{10}} & 0 & 0 \\ \frac{1}{\sqrt{40}} & \frac{5}{\sqrt{40}} & 0 \\ -1 & -5 & 4 \end{bmatrix}$ & $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Here rank is 2 & $n=3$

Index S = no of positive square terms = 2

Signature = $2S - r = 2(2) - 2 = 2$

Since $r = S < n$, the quadratic form is positive semidefinite.

: Find the transformation which will transform $4x^2+3y^2+z^2-8xy-6yz+4zx$ into sum of squares form & find the reduced form.

3. Discuss the nature of the quadratic form $x^2+4xy+6xz-y^2+2yz+4z^2$

Reduction to Canonical form through Orthogonal Transformation:-

A Quadratic form $Q = X^TAX$ can be reduced to canonical form by using an orthogonal transformation as follows

Step1 → Identify the real symmetric matrix associated with the quadratic form $Q = X^TAX$

Step2 → Determine the eigen values of A and the corresponding eigen vectors of 'A' [Since A is a symmetric matrix, A will have 'n' linearly independent eigen vectors which form an orthogonal set of vectors where n is the order set of A].

∴ A will have 'n' eigen vectors which are orthogonal and linearly independent.

Let these eigen vectors be $x_1, x_2, \cdots, x_n$

Step 3 → Normalised the eigen vectors of A are

$$\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}, \ldots, \frac{x_n}{\|x_n\|} \text{ where } \|x_i\| = \sqrt{x_1^2+x_2^2+\cdots+x_n^2}$$

Step 4 → Form the modal matrix $\hat{P}$ with these normalised eigen vectors as its columns.

$$\text{i.e., } \hat{P} = \left[\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}, \frac{x_3}{\|x_3\|} \cdots \frac{x_n}{\|x_n\|}\right]$$

[A squares matrix whose columns form an an orthogonal set of vectors becomes an orthogonal matrix]

$\therefore \hat{P}$ is Orthogonal

Hence $\hat{P}^{-1} = \hat{P}^T$

Step 5 → Form the diagonal matrix of A which is given by $D = \hat{P}^{-1} A \hat{P} = \hat{P}^T A \hat{P}$

Step 6 :→ The Canonical form of the given quadratic form is given by $Y^T D Y$ where Y is a vector

$$\text{i.e., } Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

The orthogonal transformation that transforms given Quadratic form to Canonical form $X = \hat{P}Y$.

Problems: Find the orthogonal transforms which transforms the quadratic form $Q = x^T A x$ to the canonical form $y^T D y$. where $Q = 3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_2x_3 + 2x_1x_3 - 2x_1x_2$

Sol: Given Quadratic form $Q = x^T A x = 3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_2x_3 + 2x_1x_3 - 2x_1x_2$

$= 3x_1^2 - x_1x_2 + x_1x_3 - x_2x_1 + 5x_2^2 - x_2x_3 + x_3x_1 - x_3x_2 + 3x_3^2$

The coefficient matrix A of the above quadratic form is

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (3-\lambda)[(5-\lambda)(3-\lambda) - 1] + 1[-1(3-\lambda)+1] + 1[1-(5-\lambda)] = 0$

$\Rightarrow (3-\lambda)[15 - 8\lambda + \lambda^2 - 1] + [-3+\lambda+1] + [1-5+\lambda] = 0$

$\Rightarrow (3-\lambda)[\lambda^2 - 8\lambda + 14] + [\lambda - 2] + [\lambda - 4] = 0$

$\Rightarrow 3\lambda^2 - 24\lambda + 42 - \lambda^3 + 8\lambda^2 - 14\lambda + \lambda - 2 + \lambda - 4 = 0$

$\Rightarrow -\lambda^3 + 11\lambda^2 - 36\lambda + 36 = 0$

$\Rightarrow \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

2	1	-11	36	-36
	0	2	-18	36
	1	-9	18	(0)

$(\lambda-2)(\lambda^2 - 9\lambda + 18) = 0$

$\Rightarrow (\lambda-2)(\lambda^2 - 9\lambda + 18) = 0$

$\Rightarrow (\lambda-2)(\lambda^2 - 6\lambda - 3\lambda + 18) = 0$

$\Rightarrow (\lambda-2)[\lambda(\lambda-6) - 3(\lambda-6)] = 0$

$\Rightarrow (\lambda-2)(\lambda-3)(\lambda-6) = 0$

$\Rightarrow \lambda = 2, 3, 6$

Eigen vector X_1 corresponding to eigen value $\lambda = 2$

i.e., $(A - \lambda I)X = 0 \Rightarrow (A - 2I)X = 0$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

⇒ $x_1 - x_2 + x_3 = 0$ —(i)

$x_1 + 3x_2 - x_3 = 0$ —(ii)

$x_1 - x_2 + x_3 = 0$ —(iii)

from (1) and (2) ⇒ (1)+(2)

⇒ $x_1 - x_2 + x_3 = 0$

$x_1 + 3x_2 - x_3 = 0$

$2x_2 = 0 \Rightarrow x_2 = 0$

Sub x_2 in (3)

$x_1 - 0 + x_3 = 0$

$\Rightarrow x_1 = -x_3$

If $x_3 = -1$, then $x_1 = +1$

i.e. $X_1 = \begin{bmatrix} +1 \\ 0 \\ -1 \end{bmatrix}$

for $\lambda = 3$

$\Rightarrow (A - \lambda I)x = 0 \Rightarrow (A - 3I)x = 0$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} -x_2 + x_3 = 0 \Rightarrow x_2 = x_3 \\ -x_1 + 2x_2 - x_3 = 0 \\ x_1 - x_2 = 0 \Rightarrow x_1 = x_3 \end{array}$$

$\therefore x_1 = x_2 = x_3$

$\therefore X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

for $\lambda = 4$

$\Rightarrow (A - \lambda I)x = 0$

$\Rightarrow (A - 6I)x = 0$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} -3x_1 - x_2 + x_3 = 0 \Rightarrow (1) \\ -x_1 - x_2 - x_3 = 0 \quad -(2) \\ x_1 - x_2 - 3x_3 = 0 \quad -(3) \end{array}$$

(2)+(3) ⇒ $-2x_2 - 4x_3 = 0$

⇒ $x_2 = -2x_3$

If $x_3 = 1 \Rightarrow x_2 = -2$

Sub x_2 & x_3 in (1) ⇒ $-3x_1 + 2 + 1 = 0$

⇒ $3x_1 = 3 \Rightarrow x_1 = 1$

$\therefore X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

The norm of eigen vector x_1 is $\|x_1\| = \sqrt{1+0+1} = \sqrt{2}$

$\|x_2\| = \sqrt{1+1+1} = \sqrt{3}$

$\|x_3\| = \sqrt{1+4+1} = \sqrt{6}$

Then the normalized modal matrix $\hat{P} = \left[\frac{x_1}{\|x_1\|} \quad \frac{x_2}{\|x_2\|} \quad \frac{x_3}{\|x_3\|}\right]$

$$\hat{P} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -2/\sqrt{6} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & 1/\sqrt{6} \end{bmatrix} \quad \& \quad \hat{P}^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -1/\sqrt{2} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 1/\sqrt{3} \\ \frac{1}{\sqrt{6}} & -2/\sqrt{6} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Also $\hat{P}\hat{P}^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -2/\sqrt{6} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{2}+\frac{1}{3}+\frac{1}{6} & \frac{1}{3}-\frac{2}{6} & -\frac{1}{2}+\frac{1}{3}+\frac{1}{6} \\ \frac{1}{3}-\frac{2}{6} & \frac{1}{3}+\frac{2}{6} & \frac{1}{3}-\frac{2}{6} \\ -\frac{1}{2}+\frac{1}{3}+\frac{1}{6} & \frac{1}{3}-\frac{2}{6} & \frac{1}{2}+\frac{1}{3}+\frac{1}{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore$ $\hat{P}$ is an orthogonal

i.e., $\hat{P}^{-1} = \hat{P}^T$

$\therefore$ The diagonal form of A is $D = \hat{P}^{-1} A \hat{P}$
$= \hat{P}^T A \hat{P}$

$$\text{i.e., } D = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -2/\sqrt{6} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{2}}-\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{3}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}} & \frac{3}{\sqrt{6}}+\frac{2}{\sqrt{6}}+\frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}}+\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}}+\frac{5}{\sqrt{3}}-\frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}}-\frac{10}{\sqrt{6}}-\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}}-\frac{3}{\sqrt{2}} & \frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}+\frac{3}{\sqrt{3}} & \frac{1}{\sqrt{6}}+\frac{2}{\sqrt{6}}+\frac{3}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 2/\sqrt{2} & 3/\sqrt{3} & 6/\sqrt{6} \\ 0 & 3/\sqrt{3} & -12/\sqrt{6} \\ \frac{-2}{\sqrt{2}} & 3/\sqrt{3} & 6/\sqrt{6} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{2}{2}+\frac{2}{2} & \frac{3}{\sqrt{6}}-\frac{3}{\sqrt{6}} & \frac{6}{\sqrt{12}}-\frac{6}{\sqrt{12}} \\ \frac{2}{\sqrt{6}}-\frac{2}{\sqrt{6}} & \frac{3}{3}+\frac{3}{3}+\frac{3}{3} & \frac{6}{\sqrt{18}}-\frac{12}{\sqrt{18}}+\frac{6}{\sqrt{18}} \\ \frac{2}{\sqrt{12}}-\frac{2}{\sqrt{12}} & \frac{3}{\sqrt{18}}-\frac{6}{\sqrt{18}}+\frac{3}{\sqrt{18}} & \frac{6}{6}+\frac{24}{6}+\frac{6}{6} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$\therefore$ The canonical form of the quadratic form is $Y^T D Y$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$Y^T D Y = 2y_1^2 + 3y_2^2 + 6y_3^2$$

i.e $Y^T D Y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}_{1\times 3} \begin{bmatrix} 2y_1 \\ 3y_2 \\ 6y_3 \end{bmatrix}_{3\times 1} = 2y_1^2 + 3y_2^2 + 6y_3^2$

$\therefore$ Index $s = 3$

Signature $= 3 - 0 = 3$

Nature of the quadratic form is positive definite.

Since all eigen values are positive.

2. Find the eigen vectors of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ and hence reduce the quadratic form

$Q = X^T A X = 6x^2 + 3y^2 + 3z^2 - 2yz + 4zx - 4xy$ to a sum of squares

3. Reduce the quadratic form $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$ to the canonical form by O.T

4. Reduce the Q.F $Q = 3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$ into sum of squares form by an orthogonal transformation

5 Reduce the Q.F $Q = X^T A X$ to canonical form by an orthogonal reduction and state the nature of the quadratic form

$5x^2 + 26y^2 + 10z^2 + 4yz + 14zx + 6xy$.

__Reduction To Canonical form using Lagrange's method__

__Procedure__:-

Step 1 → Take the common terms from product terms of given quadratic form

2 → Make perfect squares suitably by regrouping the terms

3 → The resulting relation gives the required canonical form

Problem: By Lagrange's reduction ... the quadratic form $x^T A x$ to sum of the squares form for

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & -2 \\ 4 & -2 & 18 \end{bmatrix}$$

Sol: Quadratic form $Q = x^T A x = [x_1 \ x_2 \ x_3] \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & -2 \\ 4 & -2 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= [x_1 \ x_2 \ x_3] \begin{bmatrix} x_1 + 2x_2 + 4x_3 \\ 2x_1 + 6x_2 - 2x_2 \\ 4x_1 - 2x_2 + 18x_3 \end{bmatrix}$

$= x_1(x_1 + 2x_2 + 4x_3) + x_2(2x_1 + 6x_2 - 2x_3) + x_3(4x_1 - 2x_2 + 18x_3)$

$= x_1^2 + 2x_1x_2 + 4x_1x_3 + 2x_1x_3 + 6x_2^2 - x_2x_3 + 4x_1x_3 - 2x_2x_3 + 18x_3^2$

$= x_1^2 + 4x_1x_2 + 8x_1x_3 + 6x_2^2 + 18x_3^2 - 4x_2x_3$

$= x_1^2 + 2x_1(x_2 + 2x_3) + 6x_2^2 + 18x_3^2 - 4x_2x_3$

$= x_1^2 + 4x_1(x_2 + 2x_3) + [2(x_2 + 2x_3)]^2 - [2(x_2 + 2x_3)]^2 + 6x_2^2 + 18x_3^2 - 4x_2x_3$

$= [x_1 + 2(x_2 + 2x_3)]^2 - 4(x_2^2 + 4x_3^2 + 4x_2x_3) + 6x_2^2 + 18x_3^2 - 4x_2x_3$

$= [x_1 + 2(x_2 + 2x_3)]^2 - 4x_2^2 - 16x_3^2 - 16x_2x_3 + 6x_2^2 + 18x_3^2 - 4x_2x_3$

$= [x_1 + 2(x_2 + 2x_3)]^2 + 2x_2^2 + 2x_3^2 - 20x_2x_3$

$= [x_1 + 2(x_2 + 2x_3)]^2 + 2[x_2^2 - 10x_2x_3] + 2x_3^2$

$= [x_1 + 2x_2 + 4x_3]^2 + 2[x_2^2 - 10x_2x_3 + (5x_3)^2] - 2(5x_3)^2 + 2x_3^2$

$= (x_1 + 2x_2 + 4x_3)^2 + 2(x_2 - 5x_3)^2 - 50x_3^2 + 2x_3^2$

$= (x_1 + 2x_2 + 4x_3)^2 + 2(x_2 - 5x_3)^2 - 48x_3^2$

$= y_1^2 + 2y_2^2 - 48y_3^2$

where $y_1 = x_1 + 2x_2 + 4x_3$, $y_2 = x_2 - 5x_3$ & $y_3 = x_3$

Index $s = 2$, signature is 1 & Matrix is Indefinite

2. Reduce the Q.F $x^2 + y^2 + 2z^2 - 2xy + 4xz + 4yz$ to canonical form by Lagrange's reduction.

MM

Sec:- ECE-A

Roll no:- 444.

Name:- K. Laxmi Prasanna.

PROPERTIES OF EIGEN VALUES

ASSIGNMENT-II

Property 1:-

The sum of the eigen values of a square matrix is equal to its trace and product of the eigen values is equal to its determinant.

ie, if A is an nxn matrix and $\lambda_1, \lambda_2, \ldots \lambda_n$ are its n eigen values, then $\lambda_1 + \lambda_2 + \ldots + \lambda_n = Tr(A)$ and $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \ldots \lambda_n = \det(A)$

Proof:-

Characteristic equation of A is $|A-\lambda I| = 0$

ie.,

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & & a_{nn}-\lambda \end{vmatrix} = 0$$

Expanding this, we get

$(a_{11}-\lambda)(a_{22}-\lambda) \ldots (a_{nn}-\lambda) - a_{12}$ (a polynomial of degree $n-2$)

$+ a_{13}$ (a polynomial of degree $n-2$) $+ \ldots + = 0$

i.e., $(-1)^n (\lambda - a_{11})(\lambda - a_{22}) \ldots (\lambda - a_{nn})$ + a polynomial of degree $(n-2) = 0$

(i.e.) $(-1)^n [\lambda^n - (a_{11} + a_{22} + \ldots + a_{nn})\lambda^{n-1}$ + a polynomial of degree $(n-2)] = 0$

+ a polynomial of degree $(n-2)$ in $\lambda = 0$.

$\therefore (-1)^n \lambda^n + (-1)^{n+1} (\text{Trace } A) \lambda^{n-1}$ + a polynomial of degree $(n-2)$ in $\lambda = 0$

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of this equation,

$$\text{Sum of the roots} = -\frac{(-1)^{n+1} Tr(A)}{(-1)^n} = Tr(A)$$

Further $|A-\lambda I| = (-1)^n \lambda^n + \ldots + a_0$

put $\lambda = 0$. Then $|A| = a_0$

$(-1)^n \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_0 = 0$.

$\Rightarrow$ product of the roots $= \dfrac{(-1)^n a_0}{(-1)^n} = a_0 = |A| = \det A$.

Hence the result.

Property 2 :-

The eigen values of a ~~diagonal~~ triangular matrix are nothing but its principle diagonal elements.

Proof :- let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ be a triangular matrix of order n.

The characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$.

i.e., $(a_{11}-\lambda)(a_{22}-\lambda) \cdots (a_{nn}-\lambda) = 0$.

$\therefore \lambda = a_{11}, a_{22}, \cdots, a_{nn}$.

Hence the eigen values of A and are $a_{11}, a_{22}, \cdots a_{nn}$ which are just the diagonal elements of A.

Property 3:-

The eigen values of a diagonal matrix are just the diagonal elements of the matrix.

Proof :-

Let $A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ be diagonal matrix.

The characteristic equation of A be $|A-\lambda I| = 0$

i.e. $$\begin{vmatrix} a_{11}-\lambda & 0 & 0 & \cdots & 0 \\ 0 & a_{22}-\lambda & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn}-\lambda \end{vmatrix} = 0.$$

$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda) \cdots (a_{nn}-\lambda) = 0.$

$\Rightarrow \lambda = a_{11}, a_{22}, \cdots, a_{nn}.$

Hence the eigen values of A are $a_{11}, a_{22}, \cdots a_{nn}$ which are just the diagonal elements of A.

Property 4 :-

A square matrix A and its transpose A^T have the same eigen values.

Proof :- We have $(A-\lambda I)^T = A^T - \lambda I^T$

$= A^T - \lambda I.$

$\therefore |(A-\lambda I)^T| = |A^T - \lambda I|$ (or)

$|A-\lambda I| = |A^T - \lambda I| \quad (\because |A^T| = |A|)$

$\therefore |A-\lambda I| = 0$ if and only if $|A^T - \lambda I| = 0$

i.e. λ is an eigen value of A if and only if λ is an eigen value of A^T.

Thus the eigen values of A and A^T are same.

Property 5 :-

If λ is an eigen value of a non-singular matrix A corresponding to the eigen vector x, then λ^{-1} is an eigen value of A^{-1} and corresponding eigen vector x itself.

(or)

Prove that the eigen values of A^{-1} are the reciprocals of the eigen values of A.

proof:- Since A is non-singular and product of the eigen values is equal to $|A|$, it follows that none of the eigen values of A is 0.

∴ If λ is an eigen value of the non-singular matrix A and X is the corresponding eigen vector $\lambda \neq 0$ and $AX = \lambda X$.

premultiplying this with A^{-1}, we get

$$A^{-1}(AX) = A^{-1}(\lambda X) \Rightarrow (A^{-1}A)X = \lambda A^{-1}X \Rightarrow IX = \lambda A^{-1}X.$$

$$\therefore X = \lambda A^{-1}X \Rightarrow A^{-1}X = \lambda^{-1}X \quad (\because \lambda \neq 0)$$

Hence by definition it follows that λ^{-1} is an eigen value of A^{-1} and X is the corresponding eigen vector.

Property 6 :-

If $\lambda_1, \lambda_2, ----, \lambda_n$ are the eigen values of a matrix A, then $k\lambda_1, k\lambda_2, ----, k\lambda_n$ are the eigen values of the matrix kA, where k is a non-zero scalar.

proof:- Let A be a square matrix of order n.

Then $|kA - \lambda kI| = |k(A-\lambda I)| = k|A-\lambda I|$ $(\because |kA| = k|A|)$

Since $k \neq 0$, therefore $|kA - \lambda I| = 0$ if and only if $|A - \lambda I| = 0$.

i.e., $k\lambda$ is an eigen value of kA if and only if λ is an eigen value of A.

Thus $k\lambda_1, k\lambda_2, ---- k\lambda_n$ are the eigen values of the matrix kA if $\lambda_1, \lambda_2, ---- \lambda_n$ are the eigen values of the matrix A.

Property 7 :-

If λ is an eigen value of A corresponding to the eigen vector x, then λ^n is eigen value of A^n corresponding to the eigen vector x.

proof :- Since λ is an eigen value of A corresponding to the eigen vector x, we have $AX = \lambda X$

premultiplying (1) by A, $A(AX) = A(\lambda X)$

(i.e.) $(AA)X = \lambda(AX)$ (i.e.) $A^2X = \lambda.\lambda X = \lambda^2 X$

Hence λ^2 is eigen value of A^2 with x itself as the corresponding eigen vector. Thus the theorem is true to $n=2$. Let the result be true for $n=k$.

Then $A^k X = \lambda^k X$.

premultiplying this by A and using $AX = \lambda X$, we get

$$A^{k+1} X = \lambda^{k+1} X.$$

which implies that λ^{k+1} is eigen value of A^{k+1} with X itself as the corresponding eigen vector. Hence by the principle of mathematical induction, the theorem is true for all positive integers n.

Property 8:-

If λ is an eigen value of the matrix A then $\lambda+k$ is an eigen value of the matrix $A+kI$.

proof:-

Let λ be an eigen value of A and X the corresponding eigen vector.

Then, by definition $AX = \lambda X$.

Now $(A+kI)X = AX + kIX = \lambda X + kX = (\lambda+k) X$.

we see that the scalar $\lambda+k$ is an eigen value of the matrix $A+kI$ and X is a corresponding eigen vector.

Property 9:-

Suppose that A and P be square matrices of order n such that P is non-singular. Then A and $P^{-1}AP$ have the same eigen values.

proof :- Consider the characteristic equation of $P^{-1}AP$. It is

$$|(P^{-1}AP) - \lambda I| = |P^{-1}AP - \lambda P^{-1}IP| \quad (\because I = P^{-1}IP)$$

$$= |P^{-1}(A - \lambda I)P|$$

$$= |P^{-1}||A - \lambda I||P|$$

$$= |A - \lambda I|, \text{ Since } |P^{-1}||P| = 1$$

Thus the characteristic polynomials of $P^{-1}AP$ and A are same. Hence the eigen values of $P^{-1}AP$ and A are same.

Property 10 :-

If A and B are n rowed square matrices and if A is invertible show that $A^{-1}B$ and BA^{-1} have same eigen values.

proof :- Given A is invertible $\Rightarrow A^{-1}$ exists

We know that if A and P are the square matrices of order 'n' such that P is non-singular, then A and $P^{-1}AP$ have the same eigen values.

Taking $A = BA^{-1}$ and $P = A$, we have

BA^{-1} and $A^{-1}(BA^{-1})A$ have the same eigen values.

i.e., BA^{-1} and $(A^{-1}B)(A^{-1}A)$ have the same eigen values.

i.e. BA^{-1} and $(A^{-1}B)I$ have the same eigen values.

or BA^{-1} and $A^{-1}B$ have the same eigen values.

Property 11 :-

If λ is an eigen value of a non-singular matrix A, then $\frac{|A|}{\lambda}$ is an eigen value of the matrix adj A.

proof :- Since λ is an eigen value of a non-singular matrix, therefore, $\lambda \neq 0$.

Also λ is an eigen value of A implies that there exists a non-zero vector X such that $AX = \lambda X$.

$\Rightarrow (\text{adj}A)AX = (\text{adj}A)(\lambda X)$

$\Rightarrow [(\text{adj}A)A]X = \lambda(\text{adj}A)X$

$\Rightarrow |A|IX = \lambda(\text{adj}A)X \quad [\because (\text{adj}A)A = |A|I]$

$\Rightarrow \frac{|A|}{\lambda}X = (\text{adj}A)X$ (or) $(\text{adj}A)X = \frac{|A|}{\lambda}X$.

Since X is a non-zero vector, therefore, from the relation (1) it is clear that $\frac{|A|}{\lambda}$ is an eigen value of the matrix adj A.

Property 12 :-

If λ is an eigen value of an orthogonal matrix then $1/\lambda$ is also an eigen value.

proof :- We know that if λ is an eigen value of a matrix A, then $1/\lambda$ is an eigen value of A^{-1}, Since A is an orthogonal matrix, therefore,

$$A^{-1} = A'$$

$\therefore \frac{1}{\lambda}$ is an eigen value of A'.

But the matrices A and A' have the same eigen values, since the determinants $|A-\lambda I|$ and $|A'-\lambda I|$ are same.

Hence $\frac{1}{\lambda}$ is also an eigen value of A.

property 13:-

If X is an eigen vector of a square matrix A, then X cannot correspond to more than one eigen value of A.

proof:- If possible, let X correspond to two eigen values λ_1 and λ_2 of A.

then, we have $AX = \lambda_1 X$ and $AX = \lambda_2 X$

$$\therefore \lambda_1 X = \lambda_2 X \Rightarrow (\lambda_2 - \lambda_1)X = 0 \Rightarrow \lambda_1 \neq \lambda_2 \quad (\because X \neq 0)$$

Hence the result.

—— X ——

therefore,

$$A^{-1} = A'$$

$\therefore \frac{1}{\lambda}$ is an eigen value of A'.

But the matrices A and A' have the same eigen values, since the determinants $|A-\lambda I|$ and $|A'-\lambda I|$ are same.

Hence $\frac{1}{\lambda}$ is also an eigen value of A.

property 13:-

If X is an eigen vector of a square matrix A, then X cannot correspond to more than one eigen value of A.

proof:- If possible, let X correspond to two eigen values λ_1 and λ_2 of A.

then, we have $AX = \lambda_1 X$ and $AX = \lambda_2 X$

$$\therefore \lambda_1 X = \lambda_2 X \Rightarrow (\lambda_2 - \lambda_1) X = 0 \Rightarrow \lambda_1 \neq \lambda_2 \quad (\because X \neq 0)$$

Hence the result.

— X —

infinite series is a puzzle for centuries. Convergence and divergence of infinite series plays an important role in engineering applications.

Sequences :-

A part of the system of natural numbers is called a set of collection of numbers. These numbers are called the elements of the set.

A Sequence is particular case of set. It so the integers 1, 2, 3 .... n, these corresponding definite numbers $a_1, a_2, \ldots, a_n, \ldots$, the set $a_1, a_2 \ldots a_n$ is called the sequence and is defined by $\{a_n\}$. Therefore in a sequence the elements are arranged in a definite order while the set is the collection of elements.

Definition:-

A sequence is a function from the domain set of natural numbers N to any set S.

Real Sequence :-

Real sequence is a function from N to R the set of real numbers; denoted by $f: N \to R$

Thus the real sequence f is set of all ordered pairs $\{n, f(n)\} = \{n = 1, 2, 3 \ldots\}$ i.e set of all pairs $(n, f(n))$ with n +ve integers

I-02

Notation:-

Since the domain of a sequence is always the same (The set of +ve integers) a may be written as $\{f(n)\}$ instead of $\{n, f(n)\}$

Examples:-

1. $\{n, 1/n\} = \{1/n\} = \{1, 1/2, 1/3, 1/4, \cdots, 1/n \cdots\}$
2. $\{n, \frac{1}{2^{n-1}}\} = \{\frac{1}{2^{n-1}}\} = \{1, 1/2, 1/2^2, \frac{1}{2^3} \cdots, \frac{1}{2^{n-1}} \cdots\}$

Constant Sequence:-

A Sequence $\{a_n\}$ is called a Constant Sequence if $a_n = c \in R \ \forall \ n \in N$

Infinite Sequence:-

Infinite sequence is a sequence in which the number of terms is infinite, and is denoted by $\{a_n\}_{n=1}^{\infty}$ on their hand finite sequence denoted by $\{a_n\}_{n=1}^{m}$ contains only a finite number of terms (m = finite)

Bounded and Unbounded Sequence:-

A Sequence $\{a_n\}$ is said to be bounded if there exists numbers m and M such that $m < a_n < M$ for every n, otherwise it is said to be unbounded

(or)

A sequence $\{a_n\}$ is said to be bounded Sequence if $\{a_n\}$ is bounded bellow and bounded above

Bounded Above Sequence:-

A Sequence $\{a_n\}$ is said to be bounded above if there exists a real number M such that $a_n \le M, \forall n \in \mathbb{N}$

Bounded Bellow Sequence:-

A Sequence $\{a_n\}$ is said to be bounded bellow ~~sso~~ if there exist a real number m such that $m \le a_n, \forall n \in \mathbb{N}$

Unbounded Sequence:-

If there exist no real number k such that $|a_n| \le k, \forall n \in \mathbb{N}$ then the sequence $\{a_n\}$ is said to be unbounded sequence

Monotonic Sequence:-

A sequence $\{a_n\}$ is said to be

(a) Monotonically increasing if $a_{n+1} \ge a_n$ for every n

i.e $a_1 \le a_2 \le a_3 \le \cdots \le a_n \le a_{n+1} \le \cdots$

(b) Monotonically Decreasing if $a_{n+1} \le a_n$ for every n

i.e $a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n \ge a_{n+1} \ge \cdots$

(c) Mononic:- If it is either monotonically increasing or Monotonically decreasing

Example:- ① $\{1/n\} = \{1, 1/2, 1/3, 1/4 \cdots\}$ bounded

Since $0 \le a_n = 1/n < 1$ and monotonically decreasing

② $\{2^n\} = \{2, 2^2, 2^3, \cdots\}$ unbounded

Since 2^n becomes ~~and~~ ~~becomes~~ larger & larger as n comes large and monotonically increasing

Limit of Sequence:-

Limit:- A number L is said to be a limit of a sequence $\{a_n\}$ and is denoted as

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_n = \lim a_n = L$$

if for every $\epsilon > 0$ $\exists$ N such that

$|a_n - L| < \epsilon$ for all $n \geq N$

Note:- A sequence may have a unique limit of may not have [more than one] limit at all

Result:- A monotonic sequence always has a limit [may be finite or infinite]

Convergence, Divergence And Oscillation of a Sequence

Convergent Sequence

A sequence $\{a_n\}$ is said to be convergent if $\lim_{n\to\infty} a_n$ is finite

(or) If $\lim_{n\to\infty} S_n = l$, then we say that the sequence $\{S_n\}$ converges to l (or) $\{S_n\}$ is convergent to limit l.

Divergent Sequence :-

1. A Sequence which is not Convergent is called a divergent sequence

2. A Sequence $\{S_n\}$ is said to divergen to infinity, if fol each $G>0$ $\exists$ $m \in \mathbb{Z}^+$ such that $S_n > G$ $\forall n>m$

If the sequence diverges to infinity, we write $S_n \to \infty$ as $n \to \infty$

3. Similarly $\overline{\mathbb{Z}}$ also

Oscillatory Sequence

If limit of a_n is not unique (oscillates finitely) of $\pm\infty$ [oscillates infinetly]

(Or) A sequence $\{a_n\}$ which is neither Convergent to a finite number nor divergent to $+\infty$ of $-\infty$ It is Called an oscillatol sequence

Examples:

(1) $\{\frac{1}{n^2}\}$ Converges since $\lim_{n\to\infty} \frac{1}{n^2} = 0$ = finite Unique

(2) $\{n\}$ divergent since $\lim_{n\to\infty} n = \infty$ = infinite

(3) $\{(-1)^n\}$ oscillates finitly,

Since $\lim_{n\to\infty} (-1)^n = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ is odd} \end{cases}$

(4) $\{(-1)^n \cdot n^2\}$ oscillates infinitly

since $\lim_{n\to\infty} (-1)^n \cdot n^2 = \pm\infty$

Note 1:- If sequence $\{a_n\}$ converges to limit 'L' 1-04

and $\{b_n\}$ converges to 'L*' then

a. $\{a_n + b_n\}$ converges to $L + L^*$

b. $\{c a_n\}$ converges to cL

c. $\{a_n \cdot b_n\}$ converges to $L \cdot L^*$

d. $\left\{\frac{a_n}{b_n}\right\}$ converges to $\frac{L}{L^*}$, provided $L^* \neq 0$

Result 2:- Every convergent sequence is bounded sequence. Converse is not true. i.e bounded sequence may not be convergent

Note 3:- A bounded monotonic sequence is convergent

Note 4:- (a) A bounded sequence which is not convergent is said to oscillate finitely

(b) A bounded sequence which is not divergent is said to oscillate infinitely.

Useful Standard Limits:-

1. (a) $\lim_{n\to\infty} \frac{1}{n} = 0$ (b) $\lim_{n\to\infty} \frac{1}{n^2} = 0$ (c) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

2. $\lim_{n\to\infty} n^{1/n} = 1$

3. $\lim_{n\to\infty} \frac{\log n}{n} = 0$

4. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$, for any x

5. $\lim_{n\to\infty} x^{1/n} = 1$ for $x>0$

6. (a) $\lim_{n\to\infty} x^n = 0$ for $|x|<1$ i.e. $-1<x<1$

(b) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for any x.

In formula (5) and (6) (b) x remains fixed, as $n\to\infty$

Infinite Series :-

Differential Equations are frequently solved by using infinite series. Fourier series, Fourier-Bessel series, etc. expansions involve infinite series. Transcendental functions [trigonometric, exponential, logarithmic, hyperbolic etc] can be expressed conveniently in terms of infinite series. Many problems that cannot be solved in terms of elementary [algebraic and transcendental] functions can also be solved in terms of infinite series.

Series:

Given a sequence of numbers $u_1, u_2, u_3 \ldots, u_n$, ... the expression

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \quad (1)$$

which is the sum of the terms of the sequence is known as a numerical series (or

Simply "Series". The numbers $u_1, u_2, \ldots$, u_n are known as the first, second, third ... n^{th} term of the series (1)

Finite Series:-

If the number of terms in a series is finite, then the series is said to be a finite series.

The finite series $u_1 + u_2 + \cdots + u_n$ can be written as $\sum_{i=1}^{n} u_i$

Infinite Series:-

If the number of terms in a series is infinite, then the series is called an infinite series

The infinite series $= u_1 + u_2 + \cdots + \cdots \infty$ can be written as $\sum_{n=1}^{\infty} u_n$

Convergent Series

An infinite series $\sum_{n=1}^{\infty} u_n$ is said to be convergent if $\sum_{i=1}^{\infty} u_n = \lim_{n\to\infty} \left(\sum_{k=1}^{\infty} u_k\right)$

$= \lim_{n\to\infty} S_n =$ finite limit value

$= S$

Here S is known as sum of the series

[Here S_n.
The sum of the first n-terms of the series is given by $u_1+u_2+\cdots+u_n$ and is denoted by S_n

$$S_1 = u_1$$
$$S_2 = u_1+u_2$$
$$S_3 = u_1+u_2+u_3$$
$$\cdots\cdots$$
$$S_n = u_1+u_2+u_3+\cdots+u_n$$]

Divergent:-

If $\lim_{n\to\infty} S_n$ does not exist (i.e $\lim_{n\to\infty} S_n = \pm\infty$) then the series (1) is said to be divergent

Oscillation:-

When $\lim_{n\to\infty} S_n$ tends to more than one limit (non-unique) or to $\pm\infty$ then the series (1) is said to be oscillatory. Thus the behaviour of convergence, divergence or oscillation of a series is the behaviour of its sequence of partial sums $\{S_n\}$

Example:-

① $1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\cdots$

Here $u_n = \frac{1}{4^{n-1}}$

$$S_n = \frac{1-r^n}{1-r} = \frac{1-(1/4)^n}{1-1/4}$$
$$= \frac{4}{3}\left(1-\frac{1}{4^n}\right)$$

$$\text{so } \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-\frac{1}{4^n}}{1-1/4}$$
$$= \frac{4}{3}\lim\left(1-1/4^n\right)$$
$$= 4/3 = \text{finite}$$

$\therefore$ Series Converges.

(2) $1^2 + 2^2 + 3^2 + \cdots + n^2 + \cdots$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6} = \infty$$

Series diverges.

(3) $7 - 4 - 3 + 7 - 4 - 3 + 7 - 4 - 3 + \cdots$

$\lim_{n\to\infty} S_n = 0$ or 7 or 3 according by the number of terms is $3m$, $3m+1$ or $3m+2$

Since the limit is not unique, Series oscillates (finitly)

Some General properties of Series:-

1. If a Series Σu_n converges to a sum 'S' then the series $c\Sigma u_n$ also converges to the sum 'cs', where c is constant

2. If the series Σu_n and Σv_n converges to sum S' & S'' respectively then the series $\Sigma(u_n+v_n)$ and $\Sigma(u_n-v_n)$ also converges to $S'+S''$ and $S'-S''$ respectively. Addition of subtraction of two series is done by termwise of term wise subtraction

3. The convergence or divergence of an infinite series remains unaffected by the addition or removal of a finite number of terms

(4) The convergence or divergence of an infinite series remains unaffected by multiplying or dividing each term by a finite number

<u>Necessary Condition for Convergence</u>:-

Necessary condition for convergence of a series Σu_n is that, its n^{th} term u_n approaches zero as n becomes infinite

i.e if series converges, then $\lim_{n\to\infty} u_n = 0$

<u>Note</u>:- The converse of the above result is not true i.e If $\lim_{n\to\infty} u_n = 0$, then the series may converge & may diverge

<u>Convergence of Geometric Series</u>:-

The Geometric series $1+r+r^2+r^3+\cdots+\infty$

is (i) Converges if $|r|<1$

(ii) Diverges if $r \geq 1$

(iii) Oscillates if $r \leq -1$

<u>proof</u>:- Let $S_n = 1+r+r^2+r^3+\cdots+r^{n-1}$

case(i) i.e; $|r|<1$

$$S_n = \frac{1-r^n}{1-r} = \frac{1}{1-r} - \frac{r^n}{1-r}$$

$$\underset{n\to\infty}{Lt} S_n = \underset{n\to\infty}{Lt}\left[\frac{1}{1-r} - \frac{r^n}{1-r}\right]$$

$$= \underset{n\to\infty}{Lt} \frac{1}{1-r} \quad \left(\because \underset{n\to\infty}{Lt} r^n = 0, |r|<1\right)$$

$\therefore \lim_{n\to\infty} S_n = \frac{1}{1-r} = $ a finite quantity

$\therefore$ The Series is Convergent

Case (ii):- i.e., $r > 1$

$$S_n = \frac{r^n - 1}{r-1} = \frac{r^n}{r-1} - \frac{1}{r-1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{r^n}{r-1} - \frac{1}{r-1}\right)$$

$\therefore S_n \to \infty$ as $n \to \infty$ ($r^n \to \infty$ as $n \to \infty$, $r > 1$)

$\therefore$ The Series is divergent

$r = 1$, the Series becomes

$1+1+1+ \cdots$

$S_n = 1+1+1+ \cdots + n$ terms

$S_n \to \infty$ as $n \to \infty$

$\therefore$ The Series is Divergent if $r \geq 1$

Case iii:- i.e., $r = -1$, the Series becomes

$1-1+1-1+1- \cdots$

$S_n = 0$ or 1 according as n is even or odd

The series oscillates between 0 and 1

If $r < -1$, let $r = -R$ so that $R > 1$

$r^n = (-R)^n = (-1)^n R^n$

$$S_n = \frac{1-r^n}{1-r} = \frac{1-(-1)^n R^n}{1+R}$$

Now $R^n \to \infty$ as $n \to \infty$ ($R > 1$)

$S_n \to -\infty$ or $+\infty$ according as n is even or odd

The series oscillates between $-\infty$ and $+\infty$

Hence, the series oscillates if $r \leq -1$

Auxiliary Series :- (P- Series)

The Series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$

(i) is Converge if $P > 1$

(ii) Diverge if $P \leq 1$

proof :- Let $S = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$

Case(i): $P > 1$

$$\frac{1}{1^p} = 1$$

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = 2/2^p = \left(\frac{1}{2}\right)^{P-1} \quad \left(\because \frac{1}{3^p} < \frac{1}{2^p}, P > 1\right)$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{4}{4^p} = \frac{2^2}{2^{2p}}$$

$$= \left(\frac{1}{2}\right)^{2(P-1)}$$

Ily $\frac{1}{8^p} + \frac{1}{9^p} + \frac{1}{10^p} + \frac{1}{11^p} + \frac{1}{12^p} + \frac{1}{13^p} + \frac{1}{14^p} + \frac{1}{15^p} < \frac{8}{8^p} = \left(\frac{1}{2}\right)^{3(P-1)}$

Adding

$$S = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \cdots + \frac{1}{n^p} + \cdots$$

$$< 1 + \left(\frac{1}{2}\right)^{P-1} + \left(\frac{1}{2}\right)^{2(P-1)} + \left(\frac{1}{2}\right)^{3(P-1)} + \cdots$$

This is clearly a geometric series where common ratio $(1/2)^{P-1}$ is less than 1 for $P > 1$

Hence the given series is convergent

Case(ii): $P = 1$

$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$

Since $1 + 1/2 > 1/2$

$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = 1/2$ and so on

Adding: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots > 1 + \frac{1}{2} + \frac{1}{2} + \cdots$

$S > 1 + \frac{1}{2} + \frac{1}{2} + \cdots$

But $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$ is a geometric series whose common ratio 1

So, the series $1/2 + 1/2 + \cdots$ is divergent

Also, by the property ① we can add and substract finite number of terms

$\therefore 1 + \frac{1}{2} + \frac{1}{2} + \cdots$ is also divergent

$\therefore S > 1 + 1/2 + 1/2 + \cdots$ must be divergent

The series is divergent if $P = 1$

case (iii) $P < 1$

$\frac{1}{2^p} > \frac{1}{2}$ and $\frac{1}{3^p} > \frac{1}{3}$ and so on

Adding $1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots > 1 + 1/2 + 1/3 + \cdots$

$S > 1 + 1/2 + 1/3 + \cdots$

By (i) the series $1 + 1/2 + \cdots$ is Divergent

The series $S = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ is divergent

The behaviour of Geometric Series & P-Series must be known

Behaviour	Geometric	P-Series
Convergent	$\lvert r \rvert < 1$	$p > 1$
Divergent	$r \geq 1$	$p \leq 1$
Oscillatory	$r \leq -1$	—

6.8 Series of Positive terms:-

Consider the series in which all the terms are +ve

Theorem:- A series of +ve terms cannot oscillate. It is either convergent or divergent

Theorem: If $u_1 + u_2 + \cdots + u_n + \cdots$ is convergent

Then $\lim_{n \to \infty} u_n = 0$

It can be used as test for divergent

If $\lim_{n \to \infty} u_n \neq 0$, then the series Σu_n is divergent

Comparison Test:-

Let Σu_n be the given series and Σv_n be the auxiliary series if $\lim_{n \to \infty} \frac{u_n}{v_n} =$ finite (non-zero) then both the series Σu_n and Σv_n will converge or diverge together

Problems

2006 Aug

1. Test the Convergent of the Series $\frac{\sqrt{n}}{n^2+1}$

Sol:- $u_n = \frac{\sqrt{n}}{n^2+1}$

$= \frac{\sqrt{n}}{n^2(1+\frac{1}{n^2})} = \frac{1}{n^{3/2}(1+\frac{1}{n^2})}$

Let $v_n = \frac{1}{n^{3/2}}$ [Taking the Difference of highest of power n in the numerated and Denominator.]

$\frac{u_n}{v_n} = \frac{1}{n^{3/2}(1+\frac{1}{n^2})} \Big/ {}^{1}/{}_{n^{3/2}}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^{3/2}}{n^{3/2}(1+{}^{1}/{}_{n^2})}$

$= \frac{1}{1+0} = 1 =$ finite value

According to Comparison test Σu_n and Σv_n Converge or diverge together.

$\Sigma v_n = \Sigma {}^{1}/{}_{n^{3/2}}$ is a p-series with $p = 3/2$

We know that p-series is Convergent if $p>1$

Therefore Σv_n Converges

Hence Σu_n Converges

2004 | 2007

2. Test the Convergence of the Series $\Sigma \sqrt{n^4+1} - \sqrt{n^4-1}$

Sol:- Here $u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$

$= \frac{(\sqrt{n^4+1} - \sqrt{n^4-1})(\sqrt{n^4+1} + \sqrt{n^4-1})}{(\sqrt{n^4+1} + \sqrt{n^4-1})}$

$$= \frac{n^4+1-n^4+1}{(\sqrt{n^4+1}+\sqrt{n^4-1})}$$

$$= \frac{2}{(\sqrt{n^4+1})+(\sqrt{n^4-1})}$$

$$= \frac{2}{n^2\left(\sqrt{1+1/n^4}+\sqrt{1-\frac{1}{n^4}}\right)} = \frac{2}{n^2\left((1+\frac{1}{n^4})^{1/2}+(1-1/n^4)^{1/2}\right)}$$

Now choosing $v_n = \frac{1}{n^2}$

$$\frac{u_n}{v_n} = \frac{2}{n^2\left(1+\frac{1}{n^4}\right)^{1/2}+\left(1-1/n^4\right)^{1/2}} \Big/ 1/n^2$$

$$= \frac{2}{\left(1+\frac{1}{n^4}\right)^{1/2}+\left(1-1/n^4\right)^{1/2}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2}{\sqrt{1+1/n^4}+\sqrt{1-1/n^4}}$$

$$= \frac{2}{1+1} = 1 \neq 0$$

According to comparison test Σu_n and Σv_n

converge or diverge together

Σv_n is a p series with $P=2>1$

A p-series with $P=2>1$

$\therefore \Sigma v_n$ is convergent

$\therefore \Sigma u_n$ also convergent

3. Discuss the Convergence of the Series.

$$\sum \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

Solution:- Here $u_n = \dfrac{1}{\sqrt{n+1}+\sqrt{n}}$

$$= \frac{1}{\sqrt{n}\left(\sqrt{1+1/n}+1\right)}$$

Taking $v_n = \dfrac{1}{\sqrt{n}}$.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)} \Big/ 1/\sqrt{n}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n}+1}$$

$$= \frac{1}{1+1} = 1/2 \neq 0 \text{ finite}$$

Thus both Σu_n and Σv_n converge or diverge together by Comparison test

But $\Sigma v_n = \Sigma \frac{1}{\sqrt{n}}$ is divergent series

Since $\Sigma \frac{1}{n^{1/2}}$ is a p-series.

A p-series is convergent if $p>1$
is divergent if $p \le 1$

Here $p = 1/2 < 1$

$\therefore \Sigma v_n$ is divergent

$\therefore \Sigma u_n$ is Divergent By Comparison test

Assignment problems

(1) Test for convergence of the series $\frac{1}{1\cdot 2}$

$$\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\cdots$$

(2) 99, 88, 83 Discuss the convergence of the series

$$\sum \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

(3) $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\cdots\infty$

(4) $\sum \sqrt{\frac{1+2^n}{1+3^n}}$

(6) $\sum_{n=1}^{\infty} \sqrt[3]{(n^3+1)}-n$

(5) $\sum \frac{1}{n^3}\left(\frac{n+2}{n+3}\right)^n$

(7) $\sum \frac{1}{\sqrt{n}} \tan\frac{1}{n}$

(8) $\sum \log(1+1/n)$

(19) $\sum \frac{1}{1+1/n}$

(20) $\sum \frac{1}{n}-\log\left(\frac{n+1}{n}\right)$

JNTU 2000

(9) $\frac{1}{1\cdot2\cdot3}+\frac{3}{2\cdot3\cdot4}+\frac{5}{3\cdot4\cdot5}+\cdots$

(21) $\sum \sin 1/n$

(22) $\sum \frac{1}{n}\sin 1/n$

(10) $\frac{1}{1+\sqrt{2}}+\frac{1}{1+2\sqrt{3}}+\frac{1}{1+3\sqrt{4}}+\cdots$

(23) $\sum \frac{n-1}{(n^2+1)\sqrt{n+2}}$

(11) $u_n=\frac{\sqrt{n+1}-\sqrt{n}}{n^p}$

(24) $\sum n\sin^2\frac{1}{n}$

(12) Test the convergence of the series whose n^{th} term is

$\sqrt{n+1}-\sqrt{n-1}$

(13) $\sum_{n=1}^{\infty}\frac{1}{n}\left[\sqrt{n^2+n+1}-\sqrt{n^2-n+1}\right]$

(14) $\frac{1}{1\cdot2}+\frac{2}{3\cdot4}+\frac{3}{5\cdot6}+\cdots+\cdots\infty$

(15) $\frac{1}{1\cdot3}+\frac{2}{3\cdot5}+\frac{3}{5\cdot7}+\cdots$

(16) $\sum_{n=0}^{\infty}\frac{2n^3+5}{4n^5+1}$

17. $\sum(\sqrt{n^2+1}-n)$

18. $\sum \frac{1}{\sqrt{n(n+1)}}$

D'Alembert's Ratio Test:-

Statement:- A positive term Series $\sum u_n$

(i) Converges if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l < 1$

(ii) Diverges if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l > 1$

proof:- Given that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$

By definition of the limit, given $\epsilon > 0$, however small, there exist a +ve integer m such that $\left| \frac{u_{n+1}}{u_n} - l \right| < \epsilon$ for $n \geq m$

i.e $l - \epsilon < \frac{u_{n+1}}{u_n} < l + \epsilon$ — (1)

(i) $l < 1$

Choose $\epsilon > 0$ such that $r = l + \epsilon < 1$

Since $l \geq 0$, $0 < r < 1$

From (1) $\frac{u_{n+1}}{u_n} < r \;\; \forall n \geq m$

$\Rightarrow u_{n+1} < r u_n \;\; \forall n \geq m$ ($\because u_n > 0$)

put $n = m, m+1, m+2, \ldots$ we get

$u_{m+1} < r u_m$

$u_{m+2} < r u_{m+1} < r^2 u_m$

$u_{m+3} < r u_{m+2} < r^3 u_m$

- - - - - -

- - - - - - etc

Adding $u_{m+1} + u_{m+2} + \cdots < u_m(r + r^2 + r^3 + \cdots)$

$$< \frac{u_r}{1-r}$$

R.H.S is Geometric Series With Common ratio $r < 1$

So, Converges.

$\therefore$ The Series $u_{m+1} + u_{m+2} + \cdots$ is Convergent

The Convergence is not affected by adding finite number of terms $u_1, u_2, \ldots, u_m$

$\therefore$ The Series $u_1 + u_2 + u_3 + \cdots + u_m + u_{m+1} + \cdots$ is Convergent

Hence Σu_n is Convergent if $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = l < 1$

~~$\therefore$ The S~~

(ii) $l > 1$

Choose $\epsilon > 0$ such that $l - \epsilon > l$

put $l - \epsilon = R$ where $R > 1$

from (1) $\frac{u_{n+1}}{u_n} > R \; \forall n \geq m \Rightarrow u_{n+1} > R u_n \; \forall n \geq m$

$(\because u_n > 0)$

put $n = m, m+1, m+2, \ldots$ we get

$u_{m+1} > R u_m$

$u_{m+2} > R u_{m+1} > R^2 u_m$

$u_{m+3} > R u_{m+2} > R^3 u_m$

$\ldots$ etc

Adding $u_{m+1} + u_{m+2} + \cdots > u_m(R + R^2 + R^3 + \cdots)$

$R + R^2 + R^3 + \cdots$ is Geometric Series which $R > 1$

The Series is Divergent

∴ $u_{m+1} + u_{m+2} + \cdots > l$ the series which [redacted]-12

Divergent

By adding finite number of terms $u_1, u_2, \ldots$
$\ldots u_m$ the series $u_1 + u_2 + \cdots + u_m + u_{m+1} + \cdots$
is Divergent

Problems:

2009 April Aug 1. $\sum_{n=1}^{\infty}\left(\frac{n^2}{2^n} + \frac{1}{n^2}\right)$ Test the Convergence of this Series

Sol:- [illegible] [illegible] [illegible] $\sum u_n$ as sum of two series $u_n = \frac{n^2}{2^n} + \frac{1}{n^2}$

$$\sum u_n = \sum \frac{n^2}{2^n} + \sum \frac{1}{n^2}$$

$$u_n = u_{n_1} + u_{n_2} \text{ i.e } \sum u_n = \sum u_{n_1} + \sum u_{n_2}$$

Where $u_{n_1} = \frac{n^2}{2^n}$

$$u_{n_1+1} = \frac{(n+1)^2}{2^{n+1}}$$

$$\frac{u_{n_1+1}}{u_{n_1}} = \frac{(n+1)^2}{2^{n+1}} \Big/ \frac{n^2}{2^n} = \frac{1}{2}\frac{(n+1)^2}{n^2}$$

$$= \frac{1}{2}\left(1+\frac{1}{n}\right)^2$$

$$\lim_{n\to\infty} \frac{u_{n_1+1}}{u_{n_1}} = \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right)^2 = \frac{1}{2} < 1$$

Therefore by D'Alembert's test $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = l < 1$

says that $\sum u_n$ is Convergent

Therefore $\sum \frac{n^2}{2^n}$ is Convergent

$\sum \frac{1}{n^2}$ is a p-series with $p=2>1$ which is Convergent

$\sum \frac{n^2}{2^n}+\frac{1}{n^2}$ is a sum of two Convergent positive term series.

Therefore it is Convergent

(2) Test the Convergence of the Series

$\frac{2}{1}+\frac{2.5}{1.5}+\frac{2.5.8}{1.5.9}+\frac{2.5.8.11}{1.5.9.13}+\cdots$

Sol: Excluding the first term $u_n=\frac{2.5.8\cdots 3n-1}{1.5.9\cdots 4n-3}$

[n=3 to ∞ since the nature of the series does not alter by adding or removing a finite no of terms]

$$u_{n+1}=\frac{2.5.8\cdots 3n+2}{1.5.9\cdots 4n+1}$$

$$\frac{u_{n+1}}{u_n}=\frac{2.5.8\cdots 3n+2}{1.5.9\cdots 4n+1}\times\frac{1.5.9\cdots 4n-3}{2.5.8\cdots 3n-1}$$

$$=\frac{3n+2}{4n+1}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{3+2/n}{4+1/n}=3/4<1$$

According to D'Alembert's Ratio test

if $\lim_{n\to\infty}\frac{u_{n+1}}{u}=l<1$ then $\sum u_n$ is Converges

∴ Given Series is Convergent

Assignment problems:

(93) 1-13

(1) Test for Convergence of the Series $\frac{x}{1\cdot2}+\frac{x^2}{3\cdot4}+\frac{x^3}{5\cdot6}+\cdots$

(2) $\frac{1}{1^k}+\frac{x}{3^k}+\frac{x^2}{5^k}+\cdots+\frac{x^{n-1}}{(2n-1)^k}+\cdots$

(3) $\sum_{n=1}^{\infty}\frac{\lfloor n}{n^n}$

(4) $\frac{1}{3}+\frac{1\cdot2}{3\cdot5}+\frac{1\cdot2\cdot3}{3\cdot5\cdot7}+\frac{1\cdot2\cdot3\cdot4}{3\cdot5\cdot7\cdot9}+\cdots$

(5) $\sum\frac{1}{x^n+x^{-n}}, x>0$ (6) $\sum\frac{2^n-2}{2^n+1}x^n, x>0$

(7)* $1+\frac{2^p}{\lfloor 2}+\frac{3^p}{\lfloor 3}+\frac{4^p}{\lfloor 4}+\cdots$

(8)** $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\cdots+\frac{x^n}{n^2+1}+\cdots\infty$

(9) $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ (10) $\sum_{n=1}^{\infty}\frac{n!\,2^n}{n^n}$ 98, 2005, 2006, 2007

(11) $\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$ (JNTU 97) (12) $x+\frac{3}{5}x^2+\frac{8}{10}x^3+\cdots+\frac{n^2-1}{n^2+1}x^n+\cdots$ 93, 94

(12) $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\cdots\infty$

(13) $x+2x^2+3x^3+4x^4+\cdots\infty$

(14) $\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+\frac{4}{1+2^4}+\cdots$

(15) If $\sum\frac{n^3+a}{2^n+a}$ is a +ve term Series test whether the Series is Convergent

(16) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n$

** (17) $\sum_{n=1}^{\infty} \left(\frac{n^2}{2^n} + \frac{1}{n^2} \right)$

** (18) Test the Convergence of the series

$$\frac{2}{1} + \frac{2 \cdot 5}{1 \cdot 5} + \frac{2 \cdot 5 \cdot 8}{1 \cdot 5 \cdot 9} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{1 \cdot 5 \cdot 9 \cdot 13} + \cdots$$

(19) $\sum_{n=1}^{\infty} \frac{n^2-n+1}{n!}$ ~~(20)~~

(20) $\frac{4}{18} + \frac{4 \cdot 12}{18 \cdot 27} + \frac{4 \cdot 12 \cdot 20}{18 \cdot 27 \cdot 36} + \cdots$

(21) $\left[\frac{1}{3}\right]^2 + \left[\frac{1 \cdot 2}{3 \cdot 5}\right]^2 + \left[\frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7}\right]^2 + \cdots$

(22) $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \cdots \infty$

OM SAI RAM OM SAI RAM OM SAI RAM

Subject: MATHEMATICS – I

UNIT: I Sequence and Series

F. Name: FACULTY OF MATHEMATICS

Department of H & S

GNIT

Expand the function $e^{-x}\sin y$ in powers of x & y. upto 3 terms.

write $f(x,y) = e^{-x}\sin y$; $f(0,0)=0$,

$f_x(x,y) = e^{-x}\sin y$; $f_x(0,0)=0$.

$f_y(x,y) = -e^{-x}\cos y$; $f_y(0,0)=-1$.

$f_{xx}(x,y) = e^{-x}\sin y$; $f_{xx}(0,0)=1$.

$f_{yy}(x,y) = e^{-x}\sin y$; $f_{yy}(0,0)=0$.

$f_{xy}(x,y) = e^{-x}\cos y$; $f_{xy}(0,0)=1$.

$f_{xxx}(x,y) = -e^{-x}\sin y$, $f_{xxx}(0,0)=0$,

$f_{xxy}(x,y) = \cos y(-e^{-x}) \quad = -1$.

$f_{xyy}(x,y) = -e^{-x}\sin y$; $=0$.

$f_{yyy}(x,y) = e^{-x}\cos y$; $=1$

∴ By Taylors theorem,

$f(x,y) = f(0,0) + x f_x(0,0) + y.f_y(0,0) + \frac{1}{2!}(x^2 f_{xx}(0,0) +$

$2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \frac{1}{3!}[x^3 f_{xxx}(0,0) + 3x^2y.$

$f_{xxy}(0,0) + 3xy^2. f_{xyy}(0,0) + y^2 f_{yyy}(0,0)]$

Raabe's Test

If Σu_n is a series of positive terms and

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = l$$

Then (i) Σu_n converges if $l>1$

(ii) Σu_n diverges if $l<1$

proof:- Let the series Σu_n be compared with

$\Sigma v_n = \Sigma \frac{1}{n^p}$ (P-Series) which converges if $p>1$

and diverges if $p\le 1$

Now $\frac{v_n}{v_{n+1}} = \frac{\frac{1}{n^p}}{1/(n+1)^p} = \left(\frac{n+1}{n}\right)^p = \left(1+\frac{1}{n}\right)^p$

$$= 1+\frac{p}{n}+\frac{p(p-1)}{\lfloor 2}\cdot\frac{1}{n^2}+\cdots$$

(i) let $\Sigma v_n = \Sigma \frac{1}{n^p}$ be convergent, so that $p>1$

The series Σu_n is convergent if $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$

$$\frac{u_n}{u_{n+1}} > 1+\frac{p}{n}+\frac{p(p-1)}{\lfloor 2}\frac{1}{n^2}+\cdots$$

$$\therefore n\left(\frac{u_n}{u_{n+1}}-1\right) > \frac{p}{n}+\frac{p(p-1)}{\lfloor 2}\frac{1}{n^2}+\cdots$$

$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) > p$ or if $l>p>1$

$\therefore \Sigma u_n$ is convergent if $l>1$

(ii) let Σv_n be divergent, so that $p\le 1$

Then Σu_n will diverge if $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$

$$\frac{u_n}{u_{n+1}} < 1 + \frac{p}{n} + \frac{p(p-1)}{\lfloor 2} \frac{1}{n^2} + \cdots$$

$$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) < p + \frac{p(p-1)}{\lfloor 2} \frac{1}{n} + \cdots$$

$$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) < p \text{ or if } l < p < 1$$

The Series Σu_n diverges if $l < 1$

Note 1:- If $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = 1$. Raabe's test fails

Note 2:- Raabe's test is applied if D'Alembert's test fails

Note 3: Bertrand's or De Morgan's test

The series whose general term u_n is Convergent & divergent according as

$$\lim_{n\to\infty}\left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n\right] > 1 \text{ or } < 1$$

Logarithmic Test:-

A positive term series Σu_n Converges or diverges if $\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} > 1$ or < 1

proof: Comparing the given series Σu_n with p-Series

$\Sigma v_n = \Sigma \frac{1}{n^p}$ which converges if $p > 1$ and diverges if $p \le 1$

Case (i) :- Let Σv_n be convergent, so that $p > 1$

Σu_n will also be convergent if $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$

$$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)^p$$

$$\Rightarrow \log \frac{u_n}{u_{n+1}} > \log\left(1+\frac{1}{n}\right)^p$$

$$= p \log\left(1+\frac{1}{n}\right) = p\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right]$$

$$\left[\text{Since } \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right]$$

i.e., $n \log \frac{u_n}{u_{n+1}} > p\left[1 - \frac{1}{2n} + \frac{1}{3n^2} - + \cdots\right]$

$$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} > p > 1 \quad (\text{Since } p > 1)$$

$$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} > 1$$

Case (ii) :- Let Σu_n be divergent so that $p \le 1$

then Σv_n also diverges if $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$

$$\log \frac{u_n}{u_{n+1}} < \log \frac{v_n}{v_{n+1}}$$

i.e., $\log \frac{u_n}{u_{n+1}} < \log\left(1+\frac{1}{n}\right)^p = p\log\left(1+1/n\right)^p$

i.e., $\log \frac{u_n}{u_{n+1}} < p\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \cdots\right]$

$$\Rightarrow n \log \frac{u_n}{u_{n+1}} < p\left[1 - \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + \cdots\right]$$

$$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} < p < 1$$

$\therefore \underset{n\to\infty}{Lt}\, n\log\frac{u_n}{u_{n+1}} < 1$

Hence the Series $\sum u_n$ converges if

$$\underset{n\to\infty}{Lt}\, n\log\frac{u_n}{u_{n+1}} > 1$$

Diverges if $\underset{n\to\infty}{Lt}\, n\log\frac{u_n}{u_{n+1}} < 1$

$\times$

Note :- The Test fails if $\underset{n\to\infty}{Lt}\, n\log\frac{u_n}{u_{n+1}} = 1$

Note :- The logarithmic test is applied if Ratio test fails and $\frac{u_n}{u_{n+1}}$ involves e and $\frac{u_n}{u_{n+1}}$ does not involve e apply Raabe's Test

Problems :- for Raabe's Test

1. Test the Convergence of the series

$$1+\frac{3x}{7}+\frac{3.6}{7.10}x^2+\frac{3.6.9}{7.10.13}x^3+\frac{3.6.9.12}{7.10.13.16}x^4+\cdots$$

Sol:- $u_n = \frac{3.6.9\ldots 3n}{7.10.13\cdots 4+3n}\, x^n$

$$u_{n+1} = \frac{3.6.9\ldots 3n+3}{7.10.13\ldots 3n+7}\, x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{3.6.9\ldots 3n.(3n+3)}{7.10.13\ldots(4+3n).(3n+7)}\cdot\frac{x^{n+1}}{x^n}\cdot\frac{7.10.13\ldots 4+3n}{3.6.9\ldots 3n}$$

$$= \frac{3n+3}{3n+7}\cdot x$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n(3+\frac{3}{n})}{n(3+7/n)} \cdot x = x$$

According to D'Alembert's Ratio test

The given series converges for $x<1$ & diverges for $x>1$ & $x=1$

if $x=1$

$$\frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}$$

[Since $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, D'Alembert's ratio test fails]

$$\frac{u_n}{u_{n+1}} - 1 = \frac{3n+7-3n-3}{3n+3}$$

$$= 4/3n+3$$

$$\Rightarrow n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{4n}{3n+3} = \frac{4}{3+3/n}$$

$$\Rightarrow \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{4}{3+3/n} = 4/3 > 1$$

According to Raabe's test the series converges

Therefore the series converges if $x \le 1$

and Diverges if $x > 1$

Q) Test the converges of the series

$$\frac{x}{1} + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot3}{2\cdot4}\cdot\frac{x^5}{5} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{x^7}{7} + \cdots \quad (x>0)$$

Sol:
$$u_n = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots 2n} \frac{x^{2n+1}}{2n+1}$$

$$u_{n+1} = \frac{1\cdot3\cdot5\cdots(2n+1)}{2\cdot4\cdot6\cdots 2n+2} \cdot \frac{x^{2n+3}}{2n+3}$$

$$\frac{u_{n+1}}{u_n} = \frac{1\cdot 3\cdot 5 \cdots 2n+1}{2\cdot 4\cdot 6 \cdots 2n+2} \cdot \frac{x^{2n+3}}{2n+3} \cdot \frac{2\cdot 4\cdot 6 \cdots 2n}{1\cdot 3\cdot 5 \cdots 2n-1} \cdot \frac{2n+1}{x^{2n+1}}$$

$$= \frac{(2n+1)}{2n+2} \cdot \frac{(2n+1)}{2n+3} \cdot x^2$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n^2(2+1/n)^2}{n^2(2+\frac{2}{n})(2+3/n)} \cdot x^2 = \lim_{n\to\infty} \frac{(2+1/n)^2}{(2+2/n)(2+3/n)} x^2$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x^2$$

According to D'Alembert's Ratio test Σu_n Converges if $x^2<1$ or $x<1$ and Diverges if $x^2>1$ or $x>1$

If $x^2=1$ (This is +ve term series) [If $l=1$ D'Alembert's Ratio test fails]

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2}$$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{4n^2+10n+6-4n^2-4n-1}{(4n^2+4n+1)}$$

$$= \frac{6n+5}{4n^2+4n}$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{6n^2+5n}{4n^2+4n}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{n^2(6+5/n)}{n^2(4+4/n)}$$

$$= \lim_{n\to\infty} \frac{(6+5/n)}{4+4/n} = 6/4 > 1$$

According to Raabe's test Σu_n Converges

Therefore Σu_n Converges if $x \le 1$

Diverges if $x > 1$

Assignment problems:-

99
1. Test for Convergence of the series $\sum \frac{4\cdot 7 \cdots (3n+1)}{1\cdot 2 \cdots n} x^n$

2. Test for the convergence of the series $1+\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\cdots\infty$

2007
3 Test the Convergence of the series

$$1+\frac{3x}{7}+\frac{3\cdot 6}{7\cdot 10}x^2+\frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3+\frac{3\cdot 6\cdot 9\cdot 12}{7\cdot 10\cdot 13\cdot 16}x^4+\cdots$$

2005, 2007
4 $\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7}+\cdots\infty \ (x>0)$

5. $1+\frac{1}{2}\cdot\frac{x^2}{4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^4}{8}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{9}{10}\frac{x^6}{12}+\cdots$

6. $\frac{1^2}{4^2}+\frac{1^2\cdot 5^2}{4^2\cdot 8^2}+\frac{1^2\cdot 5^2\cdot 9^2}{4^2\cdot 8^2\cdot 12^2}+\cdots$

7. $\frac{3^2}{6^2}+\frac{3^2}{6^2}\cdot\frac{5^2}{8^2}+\frac{3^2}{6^2}\cdot\frac{5^2}{8^2}\cdot\frac{7^2}{10^2}+\cdots$

8. $\sum \frac{2n!}{(3!)^2}x^n$

9. $x^2+\frac{2^2}{3\cdot 4}\cdot x^4+\frac{2^2\cdot 4^2}{3\cdot 4\cdot 5\cdot 6}\cdot x^6+\frac{2^2\cdot 4^2\cdot 6^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}x^8\cdots$

10. $x\log x+x^2\log 2x+x^3\log 3x+\cdots+x^n\log nx+\cdots$

96
11. $\sum \frac{4\cdot 7\cdots(3n+1)}{1\cdot 2\cdot 3\cdots n}x^n$

12. $1+\frac{(\lfloor 1)^2}{\lfloor 2}x+\frac{(\lfloor 2)^2}{\lfloor 4}x^2+\frac{(\lfloor 3)^2}{\lfloor 6}x^3+\cdots$

13. $1+\frac{2x}{\lfloor 2}+\frac{3^2x^2}{\lfloor 3}+\frac{4^3x^3}{\lfloor 4}+\frac{5^4\cdot x^4}{\lfloor 5}+\cdots$

14. $\sum \frac{(n!)^2}{(2n)!}x^n \ (x>0)$

15. $\sum \frac{1\cdot 3\cdot 5\cdots 2n-1}{2\cdot 4\cdot 6\cdots(2n+2)}$

OM SAI RAM

Problems for logarithmic Test

Statement: The positive term series Σu_n of positive series is convergent if $\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = l > 1$

Divergent if $\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} < 1$

if $l = 1$ the test fails.

(1) Test the convergence of the series

$$1+\left(\frac{1}{2}\right)^p+\left(\frac{1.3}{2.4}\right)^p+\left(\frac{1.3.5}{2.4.6}\right)^p+\cdots$$

Sol: Neglecting first term

$$u_n = \left[\frac{1.3.5\cdots 2n-1}{2.4.6\cdots 2n}\right]^p$$

$$u_{n+1} = \left[\frac{1.3.5\cdots 2n+1}{2.4.6\cdots 2n+2}\right]^p$$

$$\frac{u_n}{u_{n+1}} = \frac{1^p.3^p.5^p\cdots(2n-1)^p}{2^p.4^p.6^p\cdots(2n)^p}\cdot\frac{2^p.4^p.6^p\cdots(2n)^p(2n+2)^p}{1^p.3^p.5^p\cdots(2n-1)^p(2n+1)^p}$$

$$= \frac{(2n+2)^p}{(2n+1)^p}$$

$$\log\frac{u_n}{u_{n+1}} = \log\left(\frac{2n+2}{2n+1}\right)^p = p\log\left(\frac{2n+2}{2n+1}\right)$$

$$= p\log\left(\frac{1+1/n}{1+\frac{1}{2n}}\right)$$

$$= p\left[\log(1+1/n)-\log(1+1/2n)\right]$$

$$= p\left[\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\cdots\right)-\left(\frac{1}{2n}-\frac{1}{8n^2}+\frac{1}{24n^3}-\cdots\right)\right]$$

$$n \log \frac{u_n}{u_{n+1}} = pn\left[\frac{1}{2n} - \frac{3}{8n^2} + \frac{7}{24n^3} - \cdots\right]$$

$$= p\left[\frac{1}{2} - \frac{3}{8n} + \frac{7}{24n^2} - \cdots\right]$$

$$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = p/2$$

According to logarithmic test the given series

is Convergent if $\frac{p}{2} > 1$ i.e $p > 2$

Divergent if $\frac{p}{2} < 1$ i.e $p < 2$

when $p = 2$

$$u_n = \left(\frac{1\cdot 3\cdot 5 \cdots 2n-1}{2\cdot 4\cdot 6 \cdots 2n}\right)^2$$

$$u_{n+1} = \left(\frac{1\cdot 3\cdot 5 \cdots 2n+1}{2\cdot 4\cdot 6 \cdots 2n+2}\right)^2$$

$$\frac{u_n}{u_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^2$$

$$\log \frac{u_n}{u_{n+1}} = 2 \log\left(\frac{2n+2}{2n+1}\right)$$

$$= 2\log\left(1 + \frac{1}{2n+1}\right)$$

$$= 2\left[\frac{1}{2n+1} - \frac{1}{2(2n+1)^2} + \frac{1}{3(2n+1)^3} - \cdots\right]$$

i.e $$n \log \frac{u_n}{u_{n+1}} = 2\left[\frac{1}{2+1/n} - \frac{1}{2n(2+1/n)^2} + \frac{1}{3n^2(2+1/n)^3} - \cdots\right]$$

$$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = 2 \cdot \frac{1}{2} = 1$$

Therefore the test fails. at $p = 2$

Assignment problems:

1. Examine the Convergence of the Series

$$1+\frac{2x}{2!}+\frac{3^2x^2}{3!}+\frac{4^3x^3}{4!}+\frac{5^4x^4}{5!}+\cdots$$

2. Discuss the Convergence of the Series

$$1+\frac{x}{2}+\frac{2!}{3^2}\cdot x^2+\frac{3!}{4^3}x^3+\frac{4!}{5^4}\cdot x^4+\cdots$$

98, 2002
3. Test for the Convergence of the Series

$$x+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\frac{4^4x^4}{4!}+\cdots$$

4. Test the Convergence of the Series $\sum \frac{n^n}{e^n n!}$

5. Test the Convergence of the Series where n^{th} term is $\frac{2n!}{2^{2n}(n!)^3}$

6. Test Convergence of the Series $\sum \frac{(2n!)^3}{2^{6n}(n!)^6}$

7. Test the Convergence of the Series

$$\frac{1}{(\log 2)^2}+\frac{1}{(\log 3)^2}+\cdots+\frac{1}{(\log n)^2}+\cdots$$

8. Test the Convergence of the Series

$$1+\left(\frac{1}{2}\right)^p+\left(\frac{1\cdot3}{2\cdot4}\right)^p+\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^p+\cdots$$

OM SAI RAM

I-19

Cauchy's Root Test :-

Cauchy's General principal for Convergence :-

A necessary and sufficient condition for the existence of a limit to the sequence $\{a_n\}$ is that, for any positive integer ϵ, chosen as small as we please, there shall be a +ve number m such that $|a_n - a_m| < \epsilon$ for $n \geq m$

Cauchy's root test :-

The infinite series Σu_n is

(i) ~~the infinite series Σu_n~~

(i) Converges if $\underset{n\to\infty}{Lt} (u_n)^{1/n} = l < 1$

(ii) Diverges if $\underset{n\to\infty}{Lt} (u_n)^{1/n} = l > 1$

proof : Given $\underset{n\to\infty}{Lt} (u_n)^{1/n} = l$

By definition of limit, given $\epsilon > 0$, however small, there exists a +ve number such that

$|(u_n)^{1/n} - l| < \epsilon, \forall n \geq m$ — (I)

i.e $l - \epsilon < (u_n)^{1/n} < l + \epsilon \; \forall n \geq m$

ie $(l-\epsilon)^n < u_n < (l+\epsilon)^n, \forall n \geq m$ — (1)

(i) If $l < 1$

Now, choose $\epsilon > 0$ such that $r = l + \epsilon < 1$

Then $0 < r < 1$

From (1), we have $u_n < (l+\epsilon)^n$ for all $n \geq m$

i.e., $u_n < r^n$ for all $n \geq m$

putting $n = m, m+1, m+2, \ldots$ we get

$u_m < r^m, u_{m+1} < r^{m+1}, u_{m+2} < r^{m+2} \ldots$ and so on.

Adding all these values

$u_m + u_{m+1} + u_{m+2} + \ldots < r^m + r^{m+1} + r^{m+2} + \ldots$

The series R.H.S is geometric series whose common ratio is $r < 1$.

The series on R.H.S is Convergent

$\therefore$ The Series $u_m + u_{m+1} + u_{m+2} \ldots$ which is less than Geometric Series also Convergent

By adding finite number of terms $u_1 + u_2 + \ldots$

$\ldots + u_{m-1} + u_m + u_{m+1} + \ldots$ is

the series Σu_n is also Convergent if $r < 1$

(ii) $l > 1$

choose $\epsilon > 0$ such that $R = l - \epsilon > 1$ so that $R > 1$

from ① $u_n > R^n \ \forall n \geq m$

put $n = m, m+1, m+2 \ldots$ we get

$u_m > R^m, u_{m+1} > R^{m+1}, u_{m+2} > R^{m+2} \ldots$ and so on

Adding all these terms

$u_m + u_{m+1} + u_{m+2} + \ldots > R^m + R^{m+1} + R^{m+2} + \ldots$

The series on R.H.S Geometric Series whose Common ratio is $R > 1$

The Series on R.H.S is Divergent

The Series $u_m + u_{m+1} + u_{m+2} + \ldots$ is also Divergent

Adding finite number of terms $u_1+u_2+\cdots+u_{m-1}$, we get $u_1+u_2+\cdots+u_{m-1}+u_m+u_{m+1}+\cdots$ is Divergent

$\therefore \Sigma u_n$ is Divergent if $l>1$

Note 1:- The test fails if $\lim_{n\to\infty}(u_n)^{1/n}=l=1$ and then use Comparison test

Note 2:- This test is useful when u_n involves expression with n^{th} power

problems:-

1. Discuss the convergence of the following series (2006/09)

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots$$

Sol:- $u_n=\left[\left(\frac{n+1}{n}\right)^{n+1}-\frac{n+1}{n}\right]^{-n}$

$$(u_n)^{1/n}=\left[\left(\frac{n+1}{n}\right)^{n+1}-\frac{n+1}{n}\right]^{-1}$$

$$=\left[\frac{n+1}{n}\cdot\left(\frac{n+1}{n}\right)^{n}-\frac{n+1}{n}\right]^{-1}$$

$$=\left(\frac{n+1}{n}\right)^{-1}\left[\left(\frac{n+1}{n}\right)^{n}-1\right]^{-1}$$

$$=\frac{n}{n+1}\left[(1+1/n)^{n}-1\right]^{-1}$$

$$\lim_{n\to\infty}u_n^{1/n}=\frac{1}{e-1}<1\qquad\left[\begin{array}{l}\text{Since }\lim_{n\to\infty}(1+1/n)^{-1}=1\\ \&\left[\lim_{n\to\infty}(1+1/n)^{n}=e\right]\end{array}\right]$$

Then the series is Convergent according to Cauchy's root te.

OM SAI RAM

2. Test the Convergence of the Series $\sum \frac{(\sqrt{5}-1)^n}{n^2+1}$

Sol:- $u_n = \frac{(\sqrt{5}-1)^n}{n^2+1}$

$$(u_n)^{1/n} = \frac{\sqrt{5}-1}{(n^2+1)^{1/n}}$$

Consider $(n^2+1)^{1/n}$

$$\lim_{n\to\infty} (n^2+1)^{1/n} = y \text{ say}$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{1}{n}\right)\log(n^2+1) = \log y \quad \left(\tfrac{0}{0} \text{ form}\right) \qquad \lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{f'(n)}{g'(n)}$$

$$\Rightarrow \lim_{n\to\infty} \frac{2n}{n^2+1} = \log y \qquad \text{[By L'hospital Rule]}$$

$$\log(1+x) = \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\right]$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n}\left[n^2 - \frac{n^4}{2}\right]$$

$$\Rightarrow \log y = 0 \qquad \text{Since } \lim_{n\to\infty}\frac{\log n}{n} = 0$$

$$\Rightarrow y = 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{\log(n^2+1)}{n} = 0$$

i.e $\log y = 0$

$$\Rightarrow y = 1$$

$$\lim_{n\to\infty} (n^2+1)^{1/n} = 1$$

$$\therefore \lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{\sqrt{5}-1}{(n^2+1)^{1/n}} = \sqrt{5}-1 > 1$$

$\therefore \sum u_n$ is divergent.

3. Discuss the convergence of the series $\sum \frac{(n+1)^n x^n}{n^{n+1}}$ I-21

Sol:- $u_n = \frac{(n+1)^n x^n}{n^{n+1}}$

$$(u_n)^{1/n} = \frac{(n+1)x}{n^{1+1/n}}$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)\frac{x}{n^{1/n}}$$

$$=$$

To find $\lim_{n\to\infty} \frac{1}{n^n}$

Let $y = n^{1/n}$

$\log y = \frac{1}{n}\log n$

$$\lim_{n\to\infty} \log y = \lim_{n\to\infty} \frac{1}{n}\log n = 0$$

$$\Rightarrow \lim_{n\to\infty} \log y = 0 \Rightarrow y = 1$$

$$\therefore \lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} y = 1$$

$$\lim_{n\to\infty} u_n^{1/n} = x$$

According to Cauchy's root test

The given series is Convergent if $x<1$

& Divergent if $x>1$

when $x=1$

$$\sum u_n = \frac{(n+1)^n}{n^{n+1}}$$

Taking $\sum v_n = \frac{1}{n}$

$$\frac{u_n}{v_n} = \frac{(n+1)^n}{n^{n+1}}\cdot n = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} (1+1/n)^n = e \neq 0$$

$\therefore$ Σu_n and Σv_n converge or diverge together

Σv_n is a p-series with $p=1$ is Divergent

Σv_n is divergent. Therefore Σu_n is Divergent

Therefore the given series is Convergent if $x<1$

Divergent if $x \geq 1$

Assignment problems:-

1. Test the convergence of the given series

(a) $\Sigma \frac{2^{3n}}{3^{2n}}$ (b) $\Sigma 1/n$ (c) $\sum_{n=1}^{\infty} \frac{x^n}{(n+1)^n}$ (d) $\sum_{n=1}^{\infty} \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

2. $\frac{3x}{4} + \left(\frac{4}{5}\right)^2 x + \left(\frac{5}{6}\right)^3 x^3 + \cdots$

3. $x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \frac{5^5 x^5}{5!} + \cdots$?

4. $\frac{1}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + \left(\frac{1}{4}\right)^5 + \left(\frac{3}{4}\right)^6 + \cdots$

5. $\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3$ $\quad u_n = \left(\frac{n+1}{n+2}\right)^n x^n$

6. $1 + x/2 + \frac{x^2}{3^2} + \frac{x^3}{4^3} + \cdots$

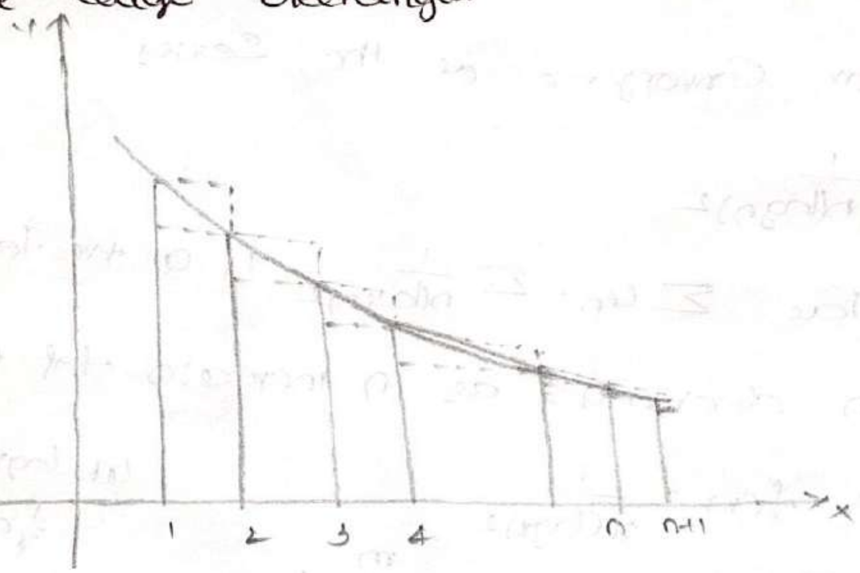
(7) $\frac{1}{4} + (2/7)^2 + \left(\frac{3}{10}\right)^3 + \cdots$ (8) $\Sigma \left(1+\frac{1}{n}\right)^{n^2}$

Cauchy's Integral Test :-

Statement:- A positive term series $f(1)+f(2)+f(3)+\cdots+f(n)+\cdots$ where $f(n)$ decreases as n increases, converges or diverges according to the integral $\int_1^{\infty} f(x)\,dx$ is finite or infinite

proof: The area bounded by the curve $y=f(x)$ between the ordinates $x=1$ and $x=n+1$ and x-axis is $\int_1^{n+1} f(x)\,dx$ [formula: Area b/w $x=a$ & $x=b$ is $\int_a^b f(x)\,dx$

This area lies between the sum of the areas of small rectangles and sum of the large rectangles



Area of the inner rectangular at $x=1$ & $x=2$ is $f(2)\times 1$ (length × breadth)

Area of the outer rectangular at $x=1$ & $x=2$ is $f(1)\times 1$

$$\therefore f(1)+f(2)+\cdots+f(n) \geq \int_1^{n+1} f(x)\,dx \geq f(2)+f(3)+\cdots+f(n+1)$$

$$S_n \geq \int_1^{n+1} f(x)\,dx \geq S_{n+1}-f(1)$$

$$S_n \geq \int_1^{n+1} f(x)\,dx \quad -①$$

$$\int_1^{n+1} f(x)\,dx \geq S_{n+1} - f(1) \quad -②$$

from ②, as $n \to \infty$, if the integral is finite

then $\lim_{n\to\infty} S_{n+1}$ is also finite

$\therefore$ So $\Sigma f(n)$ is Convergent

From ①, as $n \to \infty$, if the integral is infinite

then $\lim_{n\to\infty} S_n \to \infty$

So $\Sigma f(n)$ is divergent

Hence, the Series $f(1) + f(2) + \cdots + f(n) + \cdots$

Converges or diverges according as

$\int_1^{\infty} f(x)\,dx$ is finite or infinite

Problems:-

1. Test for Convergence of the Series

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$$

Sol:- Here $\Sigma u_n = \Sigma \frac{1}{n(\log n)^2}$ is a +ve term Series

which decreases as n increases for $n \geq 2$

$$\therefore f(x) = \frac{1}{x(\log x)^2} \qquad \text{Let } \log x = t, \ \frac{1}{x}dx = dt$$

$$\int_2^{\infty} f(x)\,dx = \lim_{m\to\infty} \int_2^{m} \frac{1}{x(\log x)^2}$$

$$= \lim_{m\to\infty} \int_2^{m} \frac{1}{t^2}dt$$

$$= \lim_{m\to\infty} \left[-t^{-1}\right]_2^{m} = \lim_{m\to\infty}\left[-\frac{1}{\log m} + \frac{1}{\log 2}\right]$$

$$= \frac{1}{\log 2} = \text{finite}$$

$\therefore$ Given Series is finite

$\therefore \sum u_n = \sum \frac{1}{n(\log n)^2}$ is Convergent

② Test the Convergence of the Series using integral test $\sum_{n=1}^{\infty} \frac{n^2+n}{n^3+\frac{3}{2}n^2+1}$

Sol:- $u_n = \frac{n^2+n}{n^3+3/2\,n^2+1}$

$u_1 = \frac{1+1}{1+3/2+1} = 4/7$

$u_2 = 5/15 = 1/3$

$u_3 = \frac{12}{41.5}$ ie $u_1 > u_2 > u_3 > \cdots$

$\therefore f(x) = \frac{x^2+x}{x^3+\frac{3}{2}x^2+1}$ Satisfies the Conditions of integral test

$$\int_1^{\infty} f(x)\,dx = \lim_{m\to\infty} \int_1^m f(x)\,dx$$

$$= \lim_{m\to\infty} \int_1^m \frac{x^2+x}{x^3+\frac{3}{2}x^2+1}\,dx$$

Let $x^3+\frac{3}{2}x^2+1 = t$

$3x^2+3x = \frac{dt}{dx}$

$(x^2+x)\,dx = \frac{1}{3}dt$

$$\therefore \int_1^{\infty} f(x)\,dx = \lim_{m\to\infty} \int_1^m \frac{1}{3}\frac{1}{t}\,dt$$

$$= \frac{1}{3}\lim_{m\to\infty} \left[\left(\log m^3+\tfrac{3}{2}\log m^2+1\right) - \log 3.5\right]$$

$= \infty$ as $m \to \infty$

$\therefore \sum u_n$ is Divergent

Assignment problems:-

1. Using the integral test, Discuss the convergent of Given series

1 (a) $\sum n e^{-n^2}$ (b) $\sum \frac{1}{n^2+1}$ (c) $\sum_{1}^{\infty} \frac{1}{n^p}$

2. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

3. Show by integral test that the series $\sum \frac{1}{n^p} (p>0)$ converges if $p>1$ and diverges if $0<p$

4. Show by Cauchy's integral test that the series $\sum_{n=1}^{\infty} \frac{1}{n(\log n)^p}$ converges if $p>1$ and diverges if $0<p\leq 1$

5. $\sum \frac{n^2}{n^3+1}$ Test the convergence by integral test

6. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

7. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

2006
8. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(\log n)^2}$

9. $\sum_{n=2}^{\infty} \frac{1}{n(\log n)}$ Test the convergence

10. Test the convergence of the series using integral test $\sum_{n=1}^{\infty} \frac{n^2+n}{(n^3+3\frac{n}{2}+1)}$

6.15 Alternating Series:-

Definition:- A series in which the terms are alternately positively and negative is called an alternating series. The series obtained from this by taking each term as positive is known as absolute series.

Absolute series Σu_n is denoted by $\Sigma|u_n|$

Leibnitz's Rule:-

Statement:- An alternating series $u_1-u_2+u_3-u_4+\cdots$ converges if (i) each term is numerically less than it's preceding term

i.e $|u_{n+1}| \leq |u_n|$ for $n \geq 1$

(ii) $\lim_{n\to\infty} u_n = 0$

proof:- Given series is $u_1-u_2+u_3+\cdots$

Given $u_1>u_2>u_3>u_4>\cdots u_n>u_{n+1}$ — ①

$$\lim_{n\to\infty} u_n = 0 \quad — ②$$

Consider the sum of $2n$ terms

$$S_{2n} = (u_1-u_2)+(u_3-u_4)+\cdots+(u_{2n-1}-u_{2n}) \rightarrow ③$$

$$\text{or } S_{2n} = u_1-(u_2-u_3)-(u_4-u_5)-\cdots-u_{2n} \rightarrow ④$$

It follows from (3) S_{2n} is positive and increasing with n

Also from (4), it follows that S_{2n} always remains less than u_1

Again $\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} (S_{2n} + u_{2n+1})$

$= \lim_{n\to\infty} S_{2n}$ $\left[\lim_{n\to\infty} u_{n+1} = 0 \text{ by } (2)\right]$

Thus $\lim_{n\to\infty} S_n$ tends to the same limit whether n is even or odd

Hence the series is convergent

Note: If $\lim_{n\to\infty} u_n \neq 0$, then $\lim_{n\to\infty} S_{2n} \neq \lim_{n\to\infty} S_{2n+1}$

So, the given series is oscillatory

Absolute and Conditional Convergence:-

Absolute Convergence :-

If $\Sigma|u_n|$ is convergent then Σu_n is said to be absolutely convergent

Conditional Convergent :-

If Σu_n is convergent and $\Sigma|u_n|$ is divergent then Σu_n is said to be conditional convergent

Note: Every absolutely convergent is convergent

Problems for Leibnitz's test :-

1. Show that series $S = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots$ Converges.

Sol:- The given series is $1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots$

$$u_1 = 1, \ u_2 = 1/3!, \ u_3 = 1/5!, \ u_4 = \frac{1}{7!}, \cdots$$

$$u_n = \frac{1}{(2n-1)!}$$

$$u_1 > u_2 > u_3 > \cdots > u_{n-1} > u_n$$

$$1 > \frac{1}{3!} > \frac{1}{5!} > \frac{1}{7!} > \cdots > \frac{1}{(2n-1)!} > \cdots$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{(2n-1)!} = 0$$

Therefore the conditions of Leibnitz's test are satisfied.

$\therefore$ The given series is Convergent

2. Test for the Convergence of the series

$\frac{1}{x} - \frac{1}{x+a} + \frac{1}{x+2a} - \frac{1}{x+3a} + \cdots$ where x and a being +ve quantity

Sol:- $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{x+(n-1)a} = \sum_{n=1}^{\infty} (-1)^{n-1} u_n$$

$$u_n = \frac{1}{x+a(n-1)}$$

$x < x+a < x+2a < \cdots$ since $a \,\&\, x > 0$

$$\frac{1}{x} > \frac{1}{x+a} > \frac{1}{x+2a} > \frac{1}{x+3a} > \cdots$$

$$\lim_{n\to\infty} \frac{1}{x+(n-1)a} = \lim_{n\to\infty} \frac{1}{x-a+na} = 0$$

Therefore according to Leibnitz's test, the given series is Convergent

Absolute and conditional convergent :-

① Discuss the absolute convergence of the series

$$\frac{1}{\sqrt{1^3+1}} - \frac{1}{\sqrt{2^3+1}}x + \frac{x^2}{\sqrt{3^3+1}} - \frac{x^3}{\sqrt{4^3+1}} + \cdots + \frac{(-1)^n x^n}{\sqrt{(n+1)^3+1}} + \cdots$$

Sol:-

$$u_1 = \frac{1}{\sqrt{1^3+1}},\ u_2 = \frac{x}{\sqrt{2^3+1}},\ u_3 = \frac{x^2}{\sqrt{3^3+1}} \cdots$$

$$u_n = \frac{(-1)^n x^n}{\sqrt{(n+1)^3+1}}$$

$$\Sigma |u_n| = \frac{1}{\sqrt{1^3+1}} + \frac{x}{\sqrt{2^3+1}} + \frac{x^2}{\sqrt{3^3+1}} + \cdots$$

$$|u_n| = \frac{x^n}{\sqrt{(n+1)^3+1}} \qquad |u_{n+1}| = \frac{x^{n+1}}{\sqrt{(n+2)^3+1}}$$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left| \frac{x^{n+1}}{\sqrt{(n+2)^3+1}} \times \frac{\sqrt{(n+1)^3+1}}{x^n} \right|$$

$$= \left| \frac{x \cdot n^{3/2}\sqrt{(1+1/n)^3+1/n^3}}{n^{3/2}\sqrt{(1+2/n)^3+1/n^3}} \right|$$

$$= \left| x \frac{\sqrt{(1+1/n)^3+1/n^3}}{\sqrt{(1+2/n)^3+1/n^3}} \right|$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = |x| \quad \text{Convergent for } |x|<1$$
$$\text{Divergent for } |x|>1$$

if $x=1$ then $|x|=1$ when $x=1$ $|u_n| = \dfrac{1}{n^{3/2}\sqrt{(1+(1/n))^3+1/n^3}}$

$$|v_n| = 1/n^{3/2},\quad \left|\frac{u_n}{v_n}\right| = \frac{n^{3/2}}{n^{3/2}\sqrt{(1+1/n)^3+1/n^3}}$$

$$= \frac{1}{\sqrt{(1+1/n)^3+1/n^3}}$$

∴ The given series is convergent absolutely in $0 < x < 1$

(2) Prove that the series $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} + \dots$ is Convergent

Sol:- $u_1 = \frac{\sin x}{1^3}$, $u_2 = \frac{\sin 2x}{2^3}$, ...

$$\Sigma |u_n| = \frac{\sin x}{1^3} + \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} + \dots$$

$$\left|\frac{\sin nx}{n^3}\right| \leq 1/n^3 \text{ for } n, \text{ since } |\sin nx| \leq 1$$

$\Sigma 1/n^3$ is a p-series with $p = 3 > 1$

A p-series with $p = 3 > 1$ is convergent

Therefore $|u_n|$ is convergent

Therefore the given alternating series is absolutely convergent

(3) Find the interval of convergence for the series $\sum_{n=1}^{\infty} (-1)^n \frac{n(x+1)^n}{2^n}$

Solution:- $$u_{n+1} = \frac{(-1)^{n+1}(n+1)(x+1)^{n+1}}{2^{n+1}}$$

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{(-1)^{n+1}(n+1)(x+1)^{n+1}}{2^{n+1}} \Big/ \frac{n\cdot(x+1)^n}{2^n}$$

$$= \left|\frac{(x+1)}{2}\cdot(1+1/n)\right|$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x+1}{2}(1+1/n)\right| = \left|\frac{x+1}{2}\right|$$

According to D'Alembert's Ratio test the series converges for $\left|\frac{x+1}{2}\right| < 1$

i.e $|x+1| < 2$

1. $-2 < x+1 < 2$

i.e $-3 < x < 1$

$x = -3$

$\Sigma (-1)^n \frac{(-2)^n}{2^n} . n = \sum_{n=1}^{\infty} . n$ which is divergent

when $x = 1$

$\sum_{n=1}^{\infty} (-1)^n . \frac{n(2)^n}{2^n} = \sum_{n=1}^{\infty} (-1)^n n$ which is oscillates infinitely

Therefore the given series is convergent when $-3 < x < 1$

(4) Test whether the following series is absolutely convergent or conditionally convergent $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \log n}$

Sol:- $\Sigma |u_n| = \sum_{n=2}^{\infty} \frac{1}{n \log n}$

This series is divergent

Alternating series $\frac{1}{2\log 2} - \frac{1}{3\log 3} + \frac{1}{4\log 4} - \frac{1}{5\log 5} + \dots$

$u_1 - u_2 + u_3 - u_4 + \dots$

$u_1 = \frac{1}{2\log 2}$, $u_n = \frac{1}{n\log n}$

$u_n = \frac{1}{(n+1)\log(n+1)}$

$(n+1)\log(n+1) > n\log n$

$\frac{1}{(n+1)\log(n+1)} < \frac{1}{n\log n}$

$u_1 > u_2 > u_3 > \dots > u_n \dots$

$\lim_{n\to\infty} \frac{1}{n\log n} = 0$ $\therefore$ According to Leibnitz's Rule

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n\log n}$ is convergent & $\sum_{n=1}^{\infty} \frac{1}{n\log n} = \Sigma |u_n|$ is divergent

$\therefore$ The given series is conditionally convergent

* Test the Convergence of the Series $\sum \frac{(\sqrt{5}-1)^n}{n^2+1}$

$u_n^{1/n} = \frac{\sqrt{5}-1}{(n^2+1)^{1/n}}$

Since $\lim_{n\to\infty} n^{1/n} = 1$

$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{\sqrt{5}-1}{(n^2+1)^{1/n}}$

$\therefore$ put $\lim_{n\to\infty} \frac{(n^2+1)^{1/n}}{n} = y$

$\lim_{n\to\infty} \log\frac{(n^2+1)^{1/n}}{n} = \log y$

$= \lim_{n\to\infty} \sqrt{5}-1 > 1$

$\Rightarrow \lim_{n\to\infty} \frac{1}{n}(\log(n^2+1)) = \log y$

$\therefore \sum u_n$ is divergent

* $\sum \frac{(n+1)^n}{n^{n+1}} x^n \rightarrow u_n = \frac{(n+1)^n}{n^{n+1}} x^n \Rightarrow \lim_{n\to\infty} u_n^{1/n} = x$

if $x=1 \Rightarrow u_n = \frac{(n+1)^n}{n^{n+1}} = \frac{n^n[1+\frac{1}{n}]^n}{n^n \cdot n} = \frac{1}{n}[1+\frac{1}{n}]^n$

$v_n = \frac{1}{n}$ $\lim_{n\to\infty} \frac{u_n}{v_n} = e \neq 0$ finite

According to C.T - diverge

* $\sum \left(\frac{n+1}{2n+5}\right)^n$ * $\sum \frac{x^{2n}}{2^n}$ * $\sum \left[1+\frac{1}{\sqrt{n}}\right]^{-n^{3/2}}$ * $\sum \frac{2^{3n}}{3^{2n}}$

* $\frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \cdots + \left(\frac{n+1}{n+2}\right)^n x^n$

Integral Test:

A positive term series $f(1)+f(2)+\cdots+f(n)+\cdots$ where $f(n)$ decreases as n increases and $\int_1^\infty f(x)\,dx = l$

~~Then if $\int_1^\infty f(x)dx$ is finite~~ then (i) ~~Then~~ l is finite, ~~then~~ given series is Conv

(ii) l is infinite, then the given series is infinite

(i) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ (ii) $\sum \frac{1}{n^2+1}$ (iii) $\sum \frac{1}{n\log n}$ (iv) $\sum \frac{n^2+n}{n^3+\frac{3}{2}n^2+1}$

if $n=2 \Rightarrow \frac{1}{2(\log 2)^2} = 5.52$

$f(x) = \frac{1}{x\log x}$

$\frac{1}{x}dx = dt$

$n=3 \Rightarrow \frac{1}{3(\log 3)^2} = 1.46$

$1/t$

$\log(\log 2)$

$\int_2^\infty \frac{1}{x(\log x)^2}dx =$ $x\to 2$, $\log 2 =$

put $\log x = t \Rightarrow \frac{1}{x}dx = dt$ $x\to\infty, t\to\infty$

$\frac{n^n}{n!}\left(\frac{1}{e}\right)^n \Rightarrow u_n^{1/n} = \frac{n}{(n!)^{1/n}} \cdot \frac{1}{e}$

$\frac{(n!)^2}{(2n)!} x^{2n}$

$u_{n+1} = \frac{((n+1)!)^2}{(2n+2)!} \cdot x^{2n+2} = n+$

$\lim_{n\to\infty} u_n^{1/n} = \frac{1}{e}\lim_{n\to\infty} \frac{n}{(n!)^{1/n}}$

UNDERTAKING

To　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　Dt:

The Principal,
Vidya Jyothi Institute of Technology,
Aziz nagar Gate,
C.B. Post,
Hyderabad – 500 075.

Sir,

I was informed by the officials of the college that my son / daughter Mr./Ms.______________________________ studying in this college bearing Roll No. __________________ in B.Tech _________ branch / MBA / _____ year ____ semester is having less attendance in this semester.

I hereby assure you that he / she will attend the classes regularly in the current semester and that in case he / she fails to do so, I have no objection if he / she is detained for want of attendance.

Name of the Parent :

Signature :

UNDERTAKING

To　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　Dt:

The Principal,
Vidya Jyothi Institute of Technology,
Aziznagar Gate,
C.B. Post,
Hyderabad – 500 075.

Sir,

I was informed by the officials of the college that my son / daughter Mr./Ms.______________________________ studying in this college bearing Roll No. __________________ in B.Tech _________ branch / MBA / _____ year ____ semester is having less attendance in this semester.

I hereby assure you that he / she will attend the classes regularly in the current semester and that in case he / she fails to do so, I have no objection if he / she is detained for want of attendance.

Name of the Parent :

Signature :

· **Ratio Test:** $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l < 1$, Σu_n converges if $l > 1$, Σu_n diverges & $l = 1$, Test fails

* $\sum \frac{(n+3)!}{3!\cdot n!\cdot 3^n}$ * $\sum_{n=1}^{\infty} \frac{1\cdot3\cdot5 \ldots (2n+1)}{2\cdot5\cdot8 \cdots 3n+2}$ * $\frac{2}{1} + \frac{2\cdot5\cdot8}{1\cdot5\cdot9} + \frac{2\cdot5\cdot8\cdot11}{1\cdot5\cdot9\cdot13} + \cdots$ $\frac{2\cdot5\cdot8\cdots(3n-1)}{1\cdot5\cdot9\cdots(4n-3)}$

* $\sum_{n=1}^{\infty} \frac{n^2}{2^n} + \frac{1}{n^2}$ * $\sum \frac{1\cdot2\cdot3\cdots n}{3\cdot5\cdot7\cdots 2n+1}$

Joseph Ludwig Raabe [1801-1859] swiss mathematician

Rabee's Test If let Σu_n be a +ve term series & $\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = l$

Then (i) Σu_n is converges, if $l > 1$
(ii) Σu_n is diverges if $l < 1$, $l = 1$ Test fails

* $\frac{2}{5}x + \frac{2\cdot4}{5\cdot8}x^2 + \frac{2\cdot4\cdot6}{5\cdot8\cdot11}x^3 + \cdots + \frac{2\cdot4\cdot6\cdots2n}{5\cdot8\cdot11\cdots3n+2}x^n + \cdots$ $(x>0)$

* $\sum \frac{1\cdot3\cdot5\cdots2n-1}{2\cdot4\cdot6\cdots2n}\cdot\frac{1}{2n+1}$ * $\sum \frac{4\cdot7\cdots(3n+1)}{1\cdot2\cdot3\cdots n} x^n$

$\frac{u_{n+1}}{u_n} = \frac{\frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}}{\frac{n^n x^n}{n!}}$

Root test

$x + \frac{2^2}{2!}x^2 + \frac{3^3}{3!}x^3 + \frac{4^4}{4!}x^4 + \cdots + \frac{n^n x^n}{n!}$

$\lim_{n\to\infty} n\left[\ldots\right]$, $u_n = \frac{n^n x^n}{n!}$

$x + \frac{2^2}{2!}x^2 + \frac{3^3}{3!}x^3$ $n\left[\frac{u_n}{u_{n+1}} - 1\right] = n\left[\frac{n^n}{(n+1)^n} - 1\right]$

$= \frac{(n+1)^n (n+1)\, x^n \cdot x}{(n+1)\, n!} \cdot \frac{n!}{n^n x^n}$

$u_n = \frac{n^{n-1} x^{n-1}}{(n-1)!}$ $u_{n+1} = \frac{(n+1)^n x^n}{n!}$ $\lim_{n\to\infty} u_n^{1/n} = 0 < 1$ Converge $\frac{(n+1)^n}{n^n}\cdot x$

$\frac{n^{n-1} x^{n-1}}{}\cdot\frac{(n+1)^n x^n}{n\cdot(n-1)!} \cdot \frac{(n-1)!}{n^{n-1} x^{n-1}}$ non-zero $\lim = \frac{n^n(1+\frac{1}{n})^n}{n^n} = x e$

$u_n^{1/n} = \left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}$

$= \frac{(n+1)^n x^n}{n\, n^n x^n} \cdot nx = \left[1+\frac{1}{n}\right]^n x = xe$

$u_n =$

$xe = 1 \Rightarrow x = 1/e$ = $u_n = \frac{n^{n-1} x^{n-1}}{(n-1)!}$ $\Rightarrow u_n = \frac{n^{n-1}}{(n-1)!}\left(\frac{1}{e}\right)^{n-1}$

$\frac{(n!)^n}{(n+1)!^n}$

$= \frac{n}{n+1}\left[\left(\frac{n+1}{n}\right)^{n}\right]^{1+1}$ $\lim_{x\to 0}\frac{1}{e-1} = \frac{1}{e-1}$ ✓

* $\sum \frac{1^2\cdot5^2\cdot9^2\cdots(4n-3)^2}{4^2\cdot8^2\cdot12^2\cdots(4n)^2}$

Cauchy's n^{th} root test: Let Σu_n be given +ve term series and $\lim_{n\to\infty} u_n^{1/n} = l$

Then (i) Σu_n is converges if $l < 1$ (ii) Σu_n diverges if $l > 1$
(iii) $l = 1 \Rightarrow$ Test fails, Then we have to use comparison test

* Discuss the convergence of the following series

$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \cdots + \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-n} + \cdots$

VIDYA JYOTHI INSTITUTE OF TECHNOLOGY

MECH-B

I B.TECH I SEM / I MID MARKS-2018-19

S.No	H.T.NO	NAME OF THE SUBJECT	ENG	M-I	EC	PPS-I	EWS	TOTAL
		Total Marks	20	20	20	20	25	
1	18911A0346	A SAI KIRAN REDDY	14	3	10	12	19	
2	18911A0347	AMETI SRI CHARAN	16	16	18	19	17	
3	18911A0348	ANIRUDH NEELAKANTAM	17	20	20	16	10	
4	18911A0349	BOBBILI CHAITANYA KRISHNA	12	9	10	5	17	
5	18911A0350	BUKYA SATHISH	12	12	18	13	20	
6	18911A0351	CHATLA SAIKIRAN	11	14	15	12	21	
7	18911A0352	CHEREDDY PRAVEENKUMAR REDDY	10	7	5	7	19	
8	18911A0353	CHINTHAKUNTLA SANDEEP KUMAR REDDY	16	17	12	19	21	
9	18911A0354	DHANE ROHITH	12	15	10	12	17	
10	18911A0355	E KIREET	16	11	11	6	18	
11	18911A0356	ELLKOLLA NAVEENKUMAR GOUD	16	16	11	13	21	
12	18911A0357	ENDURTHI NUTHAN TEJA REDDY	19	20	20	17	17	
13	18911A0358	GADIPALLY SUMEETH REDDY	19	18	18	19	22	
14	18911A0359	GANNEVARAM PAVAN KUMAR	17	13	13	9	10	
15	18911A0360	GOUNDLA ABHINESH GOUD	18	6	10	11	17	
16	18911A0361	GUGULOTHU SRIKANTH	17	5	11	9	22	
17	18911A0362	GUVALLA DAVID	19	15	16	15	23	
18	18911A0363	JUTTIGA RUDRA VARA PRASADU	17	13	15	15	24	
19	18911A0364	K LOHITH REDDY	15	10	12	11	19	
20	18911A0365	KANDUKURI VARUN KUMAR	17	12	10	9	20	
21	18911A0366	KOMPALLY SAITEJA	15	11	10	12	21	
22	18911A0367	KURUKUNTLA PAVAN SAGAR	15	16	14	17	23	
23	18911A0368	LYAGALA RAJASEKHAR	ab	ab	AB	AB	21	
24	18911A0369	MADARAM RAJESH KUMAR	20	20	19	19	23	
25	18911A0371	MALKA ESHWAR	6	1	3	5	19	
26	18911A0372	MATHANGI ANVESH KUMAR	18	11	11	14	20	
27	18911A0373	MD OMER FAROOQ	11	14	10	11	21	
28	18911A0374	MD PARVEZ	16	20	11	18	23	
29	18911A0375	MOHAMMED SHOEBUDDIN	18	14	11	8	23	
30	18911A0376	MOMULA AVINASHREDDY	11	12	4	14	21	
31	18911A0377	NAYAKAWADI RANADHEER	18	18	18	16	21	
32	18911A0378	PARIDA AVINASH	10	0	4	6	17	
33	18911A0379	PONDRETI VIDYA SAGAR	16	12	15	16	23	
34	18911A0380	PURUSHOTHAM RAKESH	7	1	1	3	18	
35	18911A0381	RAJA V KRISHNA KATHROJU	10	16	16	14	22	
36	18911A0382	RAMASWAMY MADHAVA REDDY	13	16	14	10	24	
37	18911A0383	RATHOD VENKATESH	ab	AB	AB	AB	18	
38	18911A0384	S ARAVIND REDDY	18	15	17	18	18	
39	18911A0385	SALAVATH VENKATESH	10	13	3	13	20	
40	18911A0386	SANALA BRAHMACHARY	15	20	8	13	18	
41	18911A0387	SHAIKH ANAS OSMAN SHAIKH ISRAR OSMAN	17	18	18	13	21	
42	18911A0388	SRIMANTULA SHIVA SHANKER	15	18	15	17	21	
43	18911A0389	V RAGHU	16	18	16	14	21	
44	18911A0390	VANKI ABHISHEK	14	10	10	12	19	

Unit - III

Mean Value theorems And Functions of Several Variables

Topics in III Unit

1. Rolle's theorem
2. Lagrange's Mean Value theorem
3. Cauchy's Mean Value theorem
4. Generalized Mean Value theorem
5. Functions of Several Variables
6. Functional dependence
7. Jacobian
8. Maxima and minima of two variables with Constraints and without Constraints.

(1–4: All theorems without proof)

1. Rolle's theorem

Continuous Function:-

A function $f(x)$ is said to be continuous at a point $x=a$ if Right hand limit at $x=a$ is equal to Left hand limit at $x=a$ and equal to the actual value of the function.

i.e $\lim\limits_{x\to a+0} f(x) = \lim\limits_{x\to a-0} f(x) = f(a)$

Limit of a Sequence :-

A number L is said to be a limit of a sequence $\{a_n\}$ and is denoted as

$$\lim_{n\to\infty} a_n = \lim a_n = L$$

If every $\epsilon>0$ there exists $N \ni |a_n - L| < \epsilon$ $\forall n \in N$

That is as $x \to a$, $f(x) \to L$

Continuity in an Integral :-

A function f(x) is said to be continuous on an interval I if f is continuous at each point of I

Discontinuity at a point :-

A function f(x) is said to be discontinuous at $x = a$, In other words f(x) is discontinuous at $x = a$, under any of the following circumstances.

(i) f(x) is not defined at $x = a$

(ii) $\lim_{x \to a} f(x)$ does not exist

(iii) If f(x) is defined at $x = a$

$\lim_{x \to a} f(x)$ exists but $\lim_{x \to a} f(x) \neq f(a)$

Note (1):- Every polynomial is continuous at each point

Example:- $y = x^2$ is continuous function of x

The function $y = 1/x$ is discontinuous at $x = 0$

Now $y = x^2$

$\lim_{x \to a^+} f(x) = (a)^2 = a^2$

$\lim_{x \to a^-} f(x) = (a)^2 = a^2$

$\therefore \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a) = a^2$

$\therefore y = x^2$ is a continuous function at $x = a$

Now $f(x) = y = 1/x$

$\lim_{x \to 0^+} f(x) = f(0) = \frac{1}{0} = \infty$

$$\lim_{x \to 0^-} f(x) = f(-0) = \frac{1}{0} = \infty$$

Hence f(x) is not cont at x = 0

Note (2):- A differential function is always continuous. If f(x) has a derivative in the interval $a < x < b$ then f(x) is continuous in that interval and not conversely.

i.e every continuous function not differentiable

Closed Interval Definition:-

A number x is said to belong to the closed interval [a,b] i.e $x \in [a,b]$ if $a \le x \le b$

Semi closed interval:-

$x \in (a,b]$ if $a < x \le b$

$x \in [a,b)$ if $a \le x < b$

$x \in (a,b)$ open interval if $a < x < b$

Upper and lower bounds Def:-

Suppose a function f(x) is bounds in the range $a \le x \le b$ if U has the property

i) $f(x) \le U$ for all x in the range $a \le x \le b$

(ii) $f(x) > U - \epsilon$ for at least one value of x

In the range, where $\epsilon > 0$ very small quantity. Then U is called an upper bound of f(x). If L has the property

i) $f(x) \ge L$ for all x in the range $a \le x \le$

(ii) $f(x) < L + \epsilon$ for at least one value of x in the range. Then L is called a lower bound of f(x)

Note : 3

Every continuous function in a closed interval is bounded and attains its bounds. If f(x) is continuous in the range $a \le x \le b$ and if U and L are its upper and lower bounds then there exists at least one x in the range such that f(x) = U and similarly f(x) = L for at least one x in the range $a \le x \le b$

[For Refference only]

Rolle's theorem :- 2006, 2001, 1999(S), 1997, 1996(S)

Statement :- If a function f(x) is

(i) Continuous in a closed interval [a, b]

i.e $a \le x \le b$

(ii) Derivable in the open interval (a, b)

i.e $a < x < b$

(iii) $f(a) = f(b)$

Then, there exists atleast one value of x, say c in (a, b) so that $f'(c) = 0$

proof :- Given that f(x) is continuous in (a, b)

Therefore it is bounded (Note 3)

and attains its bounds

Let U be the upper bound and L be the lower bound of f(x)

Case (i) Suppose ~~U≠L~~ U = L

~~Then either U≠L or bol~~

then by definitions of L and U

$f(x) = U = L$ for all $x \in (a,b)$ and

hence $f(x)$ is constant

$\therefore f'(x) = 0$

<u>Case (ii):</u> Suppose $U \neq L$

Then either U or L or both are different from $f(a) = f(b)$

Let $U \neq f(a)$ and $U \neq f(b)$

Then U attained for at least one value of $x = c$ in the open interval (a,b)

$\therefore f(c) = U$ and $U \neq f(a)$ of $f(b)$

$\therefore c \neq a$ and $c \neq b$

$\therefore a < c < b$

Given that f is differentiable in (a,b)

$\therefore f'(c)$ exists ~~at~~ in (a,b)

Suppose $f'(c) > 0$

$\therefore f(x) > f(c)$ in the interval of $(c, c+h)$

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} > 0$$

$\Rightarrow f(c+h) > f(c) = U$ which is a contradiction

(Since U is an upper bound by def $f(c+h) \leq h$;

$\therefore f'(c)$ cannot be greater than 0

Suppose $f'(c) < 0$

$f(x) > f(c)$ in the interval $(c-h, c)$

But $f(c) = U$ and since U is the upperbound $f(x)$ cannot be greater than $f(c)$ in [a!

$\therefore f'(c) \not< 0$ and $f'(c) \not> 0$

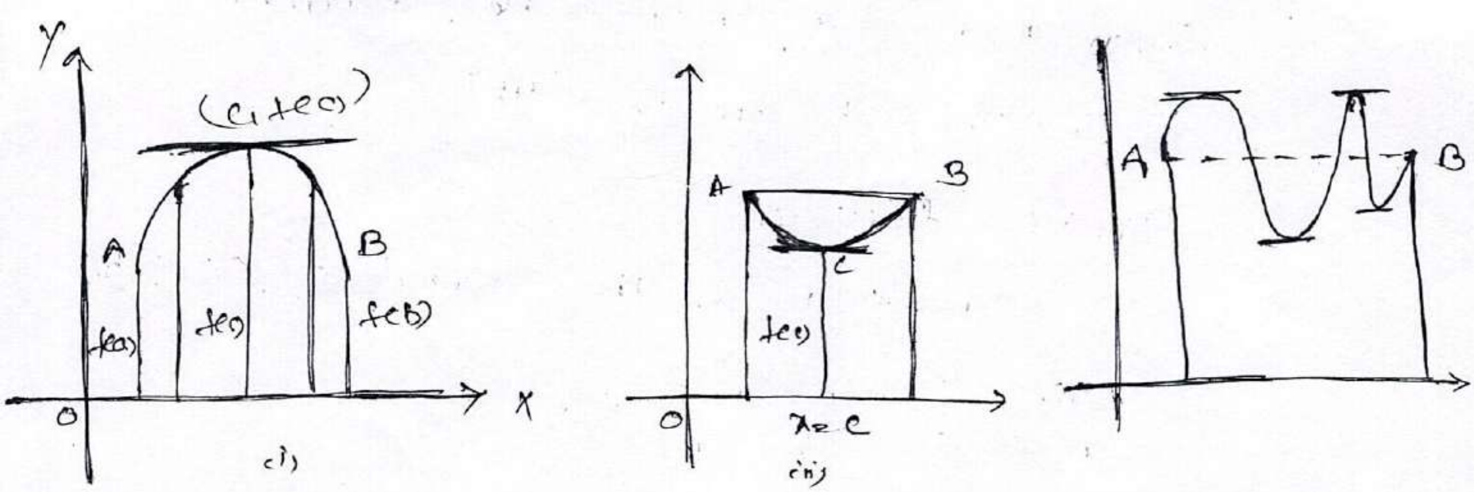
$\therefore f'(c) = 0$ and $c \in (a,b)$ i.e $a < c < b$

$\therefore \exists$ at least one value of x, say c in (a,b)

so that $f'(c) = 0$

Geometrical Interpretation:-

If a function $f(x)$ is a continuous function in a closed interval $a \le x \le b$ where $f(a) = f(b)$, then there exists atleast a point between 'a' and 'b' i.e, in the open interval $a < x < b$, say $x = c$ so that the tangent is parallel to x-axis at the point $x = c$



Note 4:- If f and g are continuous in [a,b] Then f+g, fg and f/g are also continuous in [a b]

Alternative form of Rolle's theorem

let $b = a+h$ then $c = a + \theta h$, $0 < \theta < 1$

$\Rightarrow f'(c) = f'(a + \theta h) = 0$ with $0 < \theta < 1$

Problems:-

1 1) Verify Rolle's theorem for the following function f(x) in the indicated interval.

1. $f(x) = \frac{\sin x}{e^x}$ in $(0, \pi)$

Sol:- Clearly $\sin x$ and e^x are continuous function in $[0, \pi]$

$\therefore f(x) = \frac{\sin x}{e^x}$ is continuous in $[0, \pi]$

(ii) $f(x) = \frac{\sin x}{e^x}$

$f'(x) = \frac{e^x \cos x - \sin x e^x}{e^{2x}} = \frac{\cos x - \sin x}{e^x}$ $(e^x \neq 0$

$\therefore f'(x)$ exists for all x

$\therefore f(x)$ is derivable in $(0, \pi)$

(iii) $f(0) = 0$ and $f(\pi) = \frac{\sin \pi}{e^\pi} = \frac{0}{e^\pi} = 0$

$\therefore f(0) = 0 = f(\pi) = 0$

$\therefore f(0) = f(\pi)$

$\therefore f(x)$ satisfies all conditions of Rolle's theorem

There exist atleast one value of x

say c in $(0, \pi)$ such that $f'(c) = 0$

$\Rightarrow f'(c) = 0$

$\Rightarrow \frac{\cos c - \sin c}{e^c} = 0$

$\Rightarrow \cos c = \sin c$ $\therefore c = \pi/4$

This value of c is lies on $(0, \pi)$.

2. $\ln\left[\frac{x^2+ab}{(a+b)x}\right]$ in (a,b)

Sol:- $f(x) = \log(x^2+ab) - \log(a+b) - \log x$ — ①

(i) Since $\log x$ is defined for all $x>0$ i.e $0<a<b$

$f(x)$ has unique and definite for each x in $[a\ b]$

$\therefore f(x)$ is continuous in $[a\ b]$

(ii) $f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x} = \frac{2x^2-x^2-ab}{x(x^2+ab)}$

$= \frac{x^2-ab}{x(x^2+ab)}$

$f(a) = \ln\left(\frac{a^2+ab}{(a+b)a}\right) = \ln(1) = 0$

$f(b) = \ln\left(\frac{b^2+ab}{(a+b)b}\right) = \ln(1) = 0$

$\therefore f(a) = f(b)$

$\therefore f(x)$ satisfies all the conditions of Rolle's theorem

There exists atleast one value of x say c in $(a,b) \ni f'(c) = 0$

$\therefore f'(c) = 0$

$\Rightarrow \frac{c^2-ab}{c(c^2+ab)} = 0$

$\Rightarrow \quad c^2 - ab = 0$

$\Rightarrow \quad c = \sqrt{ab} \in (a\,b)$

which lies in b/w a & b

Hence Rolle's theorem verified

(3) $f(x) = 1/x^2$ in $(-1, 1)$

Sol:- The function is not continuous at $x = 0$

The value does not exist

since $0 \in (-1, 1]$

(4) $f(x) = |x|$ in $[-1, 1]$

is continuous but not derivable at $x = 0$

Left limit of $f'(0) = \lim_{h \to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0} \frac{-h}{h} = -$

Right limit of $f'(0) = \lim_{h \to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0} \frac{h}{h} = 1$

$\therefore Lf'(c) \neq Rf'(c)$

(5) $f(x) = x^3$ in $[1, 3]$

Sol:- (i) Clearly $f(x)$ is continuous

(ii) $f'(x) = 3x^2$ which is derivable $(1, 3)$

and $f(1) = 1$, $f(3) = 27$

$f(1) \neq f(3)$

Rolle's theorem not satisfied

(4) first part

(i) $f(x) = -x, \ -1 \le x \le 0$

$= x, \ 0 \le x \le 1$

f(x) being a linear function is continuous for each value of x

Right limit $= \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$

Left limit $= \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = 0$

$\therefore$ f(x) is continuous in $(-1, 1)$

Assignment problems:-

I. Verify Rolle's theorem for the following functions

(1) $f(x) = x(x-2)e^{3x/4}$ in $(0, 2)$ Ans $c = 8/6 \in (0, 2)$

(2) $f(x) = x^{2m-1}(a-x)^{2n}$ in $(0, a)$ Ans. $c = \frac{a(2m-1)}{(2m+2n-1)}$

(3) $f(x) = x^4 + 2x^3 - 2$ in $(0, 1)$

(4) $f(x) = (x-2)\ln x$ in $(1, 2)$

2001 (5) $f(x) = e^{-x/2} x(x+3)$ in $(-3, 0)$ Ans $c = -2$

(6) $f(x) = x^2(1-x^2)$ in $0 \le x \le 1$ Ans $c = \frac{1}{\sqrt{2}}$

(7) $f(x) = x^2 - 2x$ on $[0, 2]$ Ans $c = 1$

95 (8) $f(x) = e^x(\sin x - \cos x)$ in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ Ans $c = \pi$

(9) $f(x) = x^2$ in $-1 \le x \le 1$

(10) $f(x) = (x+2)^3(x-3)^4$ in $[-2, 3]$ Ans:- $c = -2, 3, 1/2$

(11) $f(x) = 1/x^2$ in $[-1, 1]$

(12) $f(x) = 2x^3 + x^2 - 4x - 2$ in $[-\sqrt{3}, \sqrt{3}]$ Rolle's theorem is not Applicable

2003S (13) $f(x) = \tan x$ in $[0, \pi]$

Rolle's theorem is not Applicable because you will prove it

Lagrange's Mean value theorem [For Refference only]

Statement:-

If $f(x)$ is a function defined in the interva $[a\ b]$ when satisfies the following conditions

(i) $f(x)$ is continuous in $[a\ b]$

(ii) $f(x)$ is derivable in $(a\ b)$

Then there exist a value $x = c \in (a, b)$

i.e $a < c < b$

Such that $\dfrac{f(b) - f(a)}{b-a} = f'(c)$

proof:- Consider the function $\phi(x) = f(x) + Ax$

where A is constant such that $\phi(a) = \phi(b)$ — (1)

We will find the value of A

let $x = a \Rightarrow \phi(a) = f(a) + Aa$

and $x = b \Rightarrow \phi(b) = f(b) + Ab$

Since $f(x)$ is continuous in $[a\ b]$

$\phi(x)$ is also continuous in $[a\ b]$

and $f(x)$ is derivable in $(a\ b)$

then $\phi(x)$ is also derivable in $(a\ b)$

Therefore $\phi(x)$ is differentiable in (a, b)

and $\phi(a) = \phi(b)$

Therefore $\phi(x)$ satisfies all the conditions of Rolle's theorem

∴ There exists a value c in (a, b) such that

$$\phi'(c) = 0, \quad \phi'(x) = f'(x) + A$$

$$\therefore \phi'(c) = f'(c) + A = 0$$

$$\Rightarrow A = -f'(c) \quad \text{--- (2)}$$

Since from (1) $\phi(a) = \phi(b)$

$$\Rightarrow f(a) + Aa = f(b) + Ab$$

$$\Rightarrow f(a) - f(b) = Ab - Aa$$

$$\Rightarrow -(f(b) - f(a)) = A(b-a)$$

$$\Rightarrow -\frac{(f(b) - f(a))}{(b-a)} = A$$

By (2) $-f'(c) = A$

$$\Rightarrow f'(c) = \frac{(f(b) - f(a))}{(b-a)}$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b-a}$$

Hence the theorem.

Lagrange's Mean Value theorem is also known as first Mean value theorem.

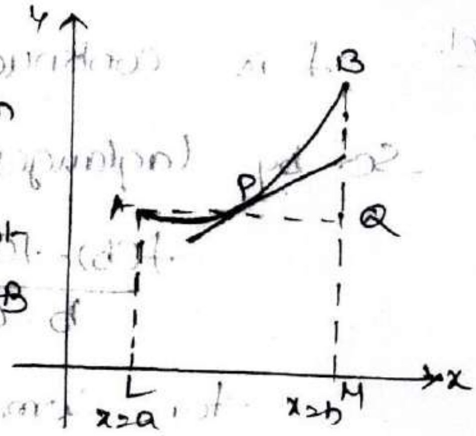
<u>Another form of Lagrange's Mean Value theorem:-</u>

If $f(x)$ is differentiable in $(a, a+h)$, there exists at least one number $\theta \in (0,1)$ such that

$$f(a+h) = f(a) + h f'(a+\theta h)$$

Geometric Interpretation:-

f(x) represents the Curve APB if the curve APB has a tangent at all points between A and B, then there exists a then point $P(x=c)$ between A, B such that the tangent at P is parallel to the chord AB

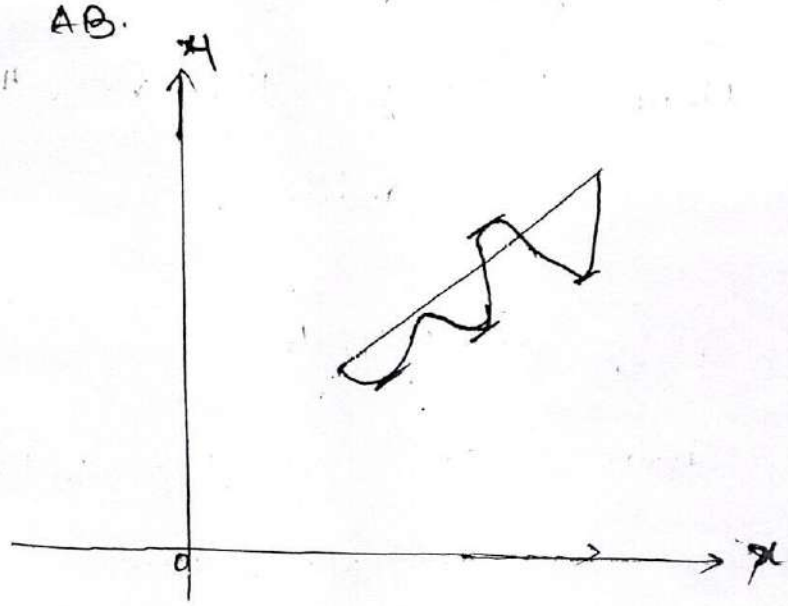


The slope of the chord $AB = \frac{BQ}{AQ}$

The slope of the tangent at $P = f'(c)$ since the chord at P is parallel to the tangent at P

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

i.e slope of the tangent to the curve at the point c is equal to the slope of the chord AB.



Verify the Lagrange's Mean value theorem

1) $f(x) = x^2$ in $(1, 5)$

Sol:- f is continuous and differentiable in (1, 5)

So by Lagrange's Mean Value theorem

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

for some c in (a, b)

Here $f(5) = 25$, $f(1) = 1$, $f'(x) = 2x$

and $f'(c) = 2c$

$$\therefore f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow 2c = \frac{25-1}{5-1} = \frac{24}{4} = 6$$

$$\Rightarrow c = 3 \in (1,5)$$

Hence Lagrange's Mean value theorem is verified.

Exam

(3) $f(x) = \cot \pi x$ at $(-1/2, 1/2)$

Sol:- Let clearly $f(x) = \cot \pi x$ at $(-1/2, 1/2)$

(i) $f(x) = \cot \pi x$ not continuous at $0 \in [-1/2, 1/2]$

Since $RHL\ f(x) = \lim_{x \to 0^+} f(x) = \cot \pi x = \cot 0 = \infty$

$LHL\ f(x) = \lim_{x \to 0^-} f(x) = \cot \pi(-0) = \cot 0 = \infty$

$\therefore$ f(x) not defined at $x = 0 \in [-1/2, 1/2]$

$\therefore$ Rolle's theorem not verified.

B since f(x) is not continuous in [-1/2, 1/2] but it ...

is derivable at (-1/2, 1/2)

because $f'(x) = -\pi \operatorname{cosec}^2 \pi x$

so it is diff in (-1/2, 1/2)

(3) Show that $\frac{h}{1+h^2} < \tan^{-1} h < h$ then $h \neq 0$ and $h > 0$

Solution:- Take $f(x) = \tan^{-1} x$ in $0 \leq x \leq h$

By Lagrange's Mean value theorem

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

since
$\begin{cases} f(a) = f(0) = \tan^{-1} 0 \\ f(b) = f(h) = \tan^{-1} h \end{cases}$

Since $f'(x) = \frac{1}{1+x^2}$, $b = h, a = 0$

Then therefore $\frac{1}{1+c^2} = \frac{\tan^{-1} h - \tan^{-1} 0}{h - 0}$

$$\Rightarrow \frac{\tan^{-1} h}{h} = \frac{1}{1+c^2}$$

$$\Rightarrow \tan^{-1} h = \frac{h}{1+c^2} \quad \text{where } 0 < c < h$$

Since $0 < c < h \Rightarrow 0^2 < c^2 < h^2$

$$\Rightarrow 1 < 1 + c^2 < 1 + h^2$$

$$\Rightarrow 1 > \frac{1}{1+c^2} > \frac{1}{1+h^2}$$

$$\Rightarrow h > \frac{h}{1+c^2} > \frac{h}{1+h^2}$$

i.e. $h > \frac{h}{1+c^2} = \tan^{-1} h > \frac{h}{1+h^2}$

(4) P.T $\pi/3 - \frac{1}{5\sqrt{3}} > \cos^{-1}\frac{3}{5} > \pi/3 - 1/8$ using Lagrange's Mean value theorem

Sol:- Let $f(x) = \cos^{-1}x$

$f'(x) = \frac{-1}{\sqrt{1-x^2}} \Rightarrow f'(c) = \frac{-1}{\sqrt{1-c^2}} \quad c \in (a,b)$

By Lagrange's Mean value theorem $f'(c) = \frac{f(b)-f(a)}{b-a}$

$$\frac{-1}{\sqrt{1-c^2}} = \frac{\cos^{-1}b - \cos^{-1}a}{b-a}$$

Since $a < c < b \Rightarrow a^2 < c^2 < b^2$

$\Rightarrow -a^2 > -c^2 > -b^2$

$\Rightarrow 1-a^2 > 1-c^2 > 1-b^2$

$\Rightarrow \sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$

$\therefore \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$

$\Rightarrow \frac{-1}{\sqrt{1-a^2}} > \frac{-1}{\sqrt{1-c^2}} > \frac{-1}{\sqrt{1-b^2}}$

$\Rightarrow \frac{-1}{\sqrt{1-a^2}} > \frac{-1}{\sqrt{1-c^2}} = \frac{\cos^{-1}b - \cos^{-1}a}{b-a} > \frac{-1}{\sqrt{1-b^2}}$

$\Rightarrow \frac{-(b-a)}{\sqrt{1-a^2}} > \cos^{-1}b - \cos^{-1}a > \frac{-(b-a)}{\sqrt{1-b^2}}$

Taking $a = 1/2$ & $b = 3/5$

$\frac{-\left(\frac{3}{5} - 1/2\right)}{\sqrt{1-1/4}} > \cos^{-1}3/5 - \cos^{-1}1/2 > \frac{-(3/5 - 1/2)}{\sqrt{1-9/25}}$

$\Rightarrow -\frac{1}{5\sqrt{3}} > \cos^{-1}3/5 - \pi/3 > -1/8$

$$\Rightarrow \pi/3 - \frac{1}{5\sqrt{3}} > \cos^{-1} 3/5 > \pi/3 - 1/8$$

Assignment problems:

Verify Lagrange's Mean value theorem (LMV)

1. $f(x) = x^2$ in $(1, 5)$
2. $f(x) = x^{2/3}$ in $(-1, 1)$
3. S.T $\frac{h}{1+h^2} < \tan^{-1} h < h$ when $h \neq 0$ & $h > 0$

2003
4. Show that for any $x \geq 0$, $1+x < e^x < 1+xe^x$
5. Show that, for any $x \in (0,1)$, $x < -\ln(1-x) < \frac{x}{1-x}$

2003
6. Calculate approximately $\sqrt[5]{245}$ by using LMV.
7. Verify Lagranges Meanvalue theorem for

$$f(x) = \begin{cases} x \sin \frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases} \quad \text{in } [-1, 1]$$

8. Show that $h < \sin^{-1} h < \frac{h}{\sqrt{(1-h^2)}}$ for $0 < h < 1$
9. Show that for $0 < a < b < 1$, $\frac{1}{1+a^2} > \frac{\tan^{-1} b - \tan^{-1} a}{b-a} > \frac{1}{1+b^2}$
10. Find 'C' of the Lagrange's theorem of

$f(x) = (x-1)(x-2)(x-3)$ on (i) $[0, 4]$
(ii) and on $[0, 0.5]$ also

11. Verify Lagrange's M.V. Theorem for $f(x) = \cos x$ in $[0, \pi/2]$
12. $f(x) = x^{1/3}$ in $[-1, 1]$

2003 S
13. $f(x) = x^3 - x^2 - 5x + 3$ in $[0, 4]$

2003 S
14. $f(x) = \log_e x$ in $[1, e)$

15. Test whether the Lagrange's M.V. theorem holds
2002 for $f(x) = x - x^3$ in $[-2, 1]$ & if so find c.

16. Show that $\frac{1}{x}\log(1+x)$ is a monotonically decreasing function of x for $x>0$.

17. If $x>0$, show that

$$x > \log(1+x) > x - \frac{1}{2}x^2$$

18. Verify L.M.V. theorem for $f(x) = e^x$ in $[0, 1]$

1996
19. Using M.V theorem, S.T $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

20. Verify L.M.V. theorem for $f(x) = x^3 - 3x - 1$ in $\left(-\frac{11}{7}, \frac{13}{7}\right)$

2006, 2007
21. P.T $\frac{\pi}{6} + \frac{\sqrt{3}}{5} < \sin^{-1}\frac{3}{5} < \frac{\pi}{6} + \frac{1}{8}$

22. For $0 < a < b$ Prove that

$$1 - \frac{a}{b} < \log\frac{b}{a} < \frac{b}{a} - 1$$ and hence S.T $\frac{1}{6} < \log\frac{6}{5} < \frac{1}{5}$

23. Apply mean value theorem P.T $x > \sin x > x - \frac{1}{6}x^3$ for $0 < x < \pi/2$

24. Verify L.M.V theorem $f(x) = 22x^2 - 7x + 10$, $a = 2, b = 5$

2007, 1998, 95
25. Prove that $\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}\frac{3}{5} > \frac{\pi}{3} - \frac{1}{8}$

1997
26. Verify L.M.V. theorem for $f(x) = x^3 - 2x^2$ in $(2, 5)$

Cauchy's Mean Value theorem:-

2007 Aug, 2000 S, 1994 S

Statement:- [For Refference]

Let $f(x)$ and $g(x)$ be two functions, which satisfy the following properties.

(i) $f(x)$, $g(x)$ are continuous in $[a,b]$

(ii) $f(x)$, $g(x)$ are differentiable in (a,b), $g'(x) \neq 0$ for all x in (a,b)

Then there exists a 'c' in (a,b) such that-

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

proof:- Consider $\phi(x) = f(x) + A\,g(x)$

Where A is constant such that

$\phi(a) = \phi(b)$ - I

$\phi(a) = f(a) + A\,g(a)$ - (1)

$\phi(b) = f(b) + A\,g(b)$ - (2)

Since $\phi(a) = \phi(b)$

$\Rightarrow f(a) + A g(a) = f(b) + A g(b)$

$\Rightarrow f(a) - f(b) = A g(b) - A g(a)$

$\Rightarrow -(f(b) - f(a)) = A(g(b) - g(a))$

$\Rightarrow -A = \dfrac{f(b)-f(a)}{g(b)-g(a)}$ - (3)

Since $f(x), g(x)$ are continuous then $\phi(x)$ is continuous in $[a,b]$

and $f(x), g(x)$ are differentiable in (a,b)

then $\phi(x)$ is differentiable in (a,b)

And $\phi(a) = \phi(b)$. (By Choosing)

$\therefore$ $\phi(x)$ satisfies all conditions of Rolle's theorem

Then there exists a value c in (a, b)

such that $\phi'(c) = 0$ — (4)

But $\phi'(x) = f'(x) + A\, g'(x)$

$$\phi'(c) = f'(c) + A\, g'(c)$$

$$\Rightarrow 0 = f'(c) + A\, g'(c) \quad \text{By (4)}$$

$$\Rightarrow -A\, g'(c) = f'(c)$$

$$\Rightarrow -A = \frac{f'(c)}{g'(c)} \quad \text{— (5)}$$

from (3) and (5) we get

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Hence the theorem

Another form of Cauchy's Mean Value theorem:-

If $f(x)$ and $g(x)$ are continuous in $[a, a+h]$ and $f(x)$ and $g(x)$ are differentiable in $(a, a+h)$ and $g'(x) \neq 0$ in $[a, a+h]$

There exists a 'θ' $\ni 0 < \theta < 1$

and $$\frac{f(a+h) - f(a)}{g(a+h) - g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}$$

$= \times =$

Problems:- Verify Cauchy's Mean Value theorem for the functions

(i) $f(x) = x^4$, $g(x) = x^2$ in the interval $[a, b]$

(ii) $f(x) = \ln(x)$, $g(x) = \frac{1}{x}$ in $[1, e]$

Sol:- (i) Cauchy's Mean value theorem states that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

(i) $f(x) = x^4$, $g(x) = x^2$, $[a\ b]$

$f'(x) = 4x^3$, $g'(x) = 2x$ in $(a\ b)$

By Cauchy's Mean Value theorem

$$\frac{b^4 - a^4}{b^2 - a^2} = \frac{4c^3}{2c}$$

$$\Rightarrow 2c^2 = b^2 + a^2$$

$$\Rightarrow c = \frac{1}{\sqrt{2}}\sqrt{b^2 + a^2} \in (a, b)$$

(ii) $f(x) = \ln(x)$, $g(x) = \frac{1}{x}$ $[1, e]$

$f'(x) = \frac{1}{x}$ $\qquad g'(x) = -\frac{1}{x^2}$

By CMV theorem $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$

$$\Rightarrow \frac{1/c}{-1/c^2} = \frac{\log b - \log a}{1/b - 1/a}$$

$$\Rightarrow -c = \log b/a \cdot \left(\frac{ab}{a-b}\right)$$

$$\Rightarrow -c = \log e \cdot \frac{e}{1-e}$$

$$\Rightarrow -c = \frac{e}{1-e} \qquad \log_e e = 1$$

$$\Rightarrow c = \frac{e}{e-1} \in (1, e)$$

2. Show that $\sqrt{\frac{1-x}{1+x}} < \frac{\ln(1+x)}{\sin^{-1}x} < 1$ if $0 < x < 1$

$f(x) = \ln(1+x)$, $g(x) = \sin^{-1}x$

Sol:- Apply CMV theorem to functions $f(x) = \ln(1+x)$

and $g(x) = \sin^{-1}x$ in $[a,b]$ then by CMV theorem

$$f'(x) = \frac{1}{1+x}, \quad g'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f'(c) = \frac{1}{1+c} \quad g'(c) = \frac{1}{\sqrt{1-c^2}}$$

such that $\exists\, c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{\log(1+b)-\log(1+a)}{\sin^{-1}b - \sin^{-1}a}$$

$$\Rightarrow \frac{\frac{1}{1+c}}{\frac{1}{\sqrt{1-c^2}}} = \frac{\log(1+b)-\log(1+a)}{\sin^{-1}b \; \sin^{-1}a}$$

$$\Rightarrow \sqrt{\frac{1-c}{1+c}} = \frac{\log(1+b)-\log(1+a)}{\sin^{-1}b - \sin^{-1}a}$$

Assignment problems: Verify C.M.V. theorem

1. Find c of Cauchy's Mean value theorem $f(x) = \sqrt{x}$ & $g(x) = \frac{1}{\sqrt{x}}$ in $[a\,b]$
2. $f(x) = x^2$ & $g(x) = x^3$ in $[1,2]$
3. $f(x) = e^{x}$ & $g(x) = e^{-x}$ in $(a\,b)$
4. $f(x) = \frac{1}{x^2}$ & $g(x) = \frac{1}{x}$ on $[a\,b]$
5. (5) $f(x) = \sin x$ & $g(x) = \cos x$ on $[0, \pi/2]$

Gr. Generalized Mean Value theorem:-

Higher Mean value theorem or Taylor's development of a function in a finite form with Lagrange's form of remainder.

Taylor's theorem with Lagrange's form of Remainder

Statement:-

Suppose $f(x)$ and all its derivatives up to $(n-1)^{th}$ order are continuous in the closed interval $[a\ b]$ and $f^n(x)$ exists for all $x \in (a, b)$ then there exists a value c, $a < c < b$ such that

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(b-a)^n}{n!} f^n(c)$$

(or). $$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^n(a+\theta h)$$

where $b-a=h$, and $0<\theta<1$

proof:- Consider the function $\phi(x)$ as follows

$$\phi(x) = F(x) - \left(\frac{b-x}{b-a}\right)^n F(a)$$

where $$F(x) = f(b) - f(x) - (b-x) f(x) - \frac{(b-x)^2}{2!} f''(x) - \cdots \frac{(b-x)^{n-1}}{(n-1)!} f^{n-1}(x)$$

Given that $f(x)$ and its derivatives upto $(n-1)^{th}$ order are continuous in $[a\ b]$

Therefore $\phi(x)$ is continuous in $[a\ b]$ and $f^n(x)$ exists in (a, b)

Therefore $\phi(x)$ is differentiable in (a, b) $\phi(a)=0$, $\phi(b)=0$

Therefore $\phi(x)$ is satisfies all conditions of Rolles theorem

According Rolle's theorem

$\phi'(x) = 0$ for all $x \in (a,b)$ — ①

$\phi(x) = 0$ for all $x \in (a,b)$

$\Rightarrow F'(x) + \frac{n(b-x)^{n-1}}{(b-a)^n} F(a) = 0$ by ①

for all $x \in (a,b)$

$$F'(x) = -f'(x) - [(b-x)f''(x) - f'(x)] - \frac{1}{2!}[(b-x)^2 f'''(x)$$

$$- 2(b-x)f''(x)] - \cdots$$

$$\cdots - \frac{1}{(n-1)!}\left[(b-x)^{n-1} f^n(x) - (n-1)(b-x)^{n-2} f^{n-1}(x)\right]$$

$$= -\frac{1}{(n-1)!}(b-x)^{n-1} f^n(x) - \frac{(b-x)^{n-1}}{(n-1)!} f^n(x) +$$

$$+ \frac{n(b-x)^{n-1}}{(b-a)^n} F(a) = 0$$

$$\Rightarrow f(b) - f(a) - (b-a)f'(a) - \frac{(b-a)^2}{2!} f''(a)$$

$$- \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) = \frac{(b-a)^n}{n!} f^n(x)$$

for all $x \in (a,b)$ i.e $x = a + \theta h$

where $0 < \theta$ and $h = b-a$

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots$$

$$+ \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^n}{n!} f^n(x)$$

put $b - a = h$

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) +$$

$$+ \frac{h^n}{n!} f^n(a+\theta h) \quad — ②$$

$\frac{(b-a)^n}{n!} f^n(a+\theta h)$ is known as Lagrange's form of remainder after the n^{th} term and written as R_n

Maclaurins theorem with Lagrange's Form of Remainder:-

put $a=0$, $h=x$ in Taylor's theorem in the formula

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \cdots + \frac{x^{n-1}}{\lfloor n-1} f^{n-1}(0) + \frac{x^n}{\lfloor n} f^n(\theta x)$$

$(0<\theta<1)$

Taylor's theorem with Cauchy's form of Remainder:-

Statement:- Let a function $f(x)$ be such that

(i) $f(x), f'(x), f''(x), \ldots, f^{n-1}(x)$ are continuous in the closed interval $a \le x \le a+h$

(ii) $f^n(x)$ exists in the open interval $a<x<a+h$

then there exists atleast one number θ between 0 and 1 such that $f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) +$

$$\cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{(n-1)!} (1-\theta)^{n-1} f^n(a+\theta h) \quad \text{--- (3)}$$

Maclaurins theorem with Cauchy's form of Remainder:

put $a=0$, $h=x$ in previous ③, we get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{\lfloor n-1} f^{n-1}(0)$$

$$+ \frac{x^n}{(n-1)!} (1-\theta)^{n-1} f^n(\theta x), \quad (0<\theta<1)$$

The $(n+1)^{th}$ term $\frac{x^n}{\lfloor n-1} (1-\theta)^{n-1} f^n(\theta x)$ is called Cauchy form of remainder in Maclaurins development of $f(x)$ in the interval $[0, x]$

Taylor's Series:

The remainder $R_n \to 0$ as $n \to \infty$

Then the Taylor's is given by

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{\lfloor 2} f''(a) + \cdots + \frac{h^n}{\lfloor n} f^n(a) + \cdots$$

Maclaurin's Series:

put $a=0$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \cdots$$

Some Expansions:-

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^5}{5} + \cdots$$

$$\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2} x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} x^3 + \cdots$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{1\cdot 2} x^2 + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3} x^3 + \cdots$$

1. Verify Taylor's theorem for $f(x) = x^3 - 3x^2 + 2x$ in $[0, 1/2]$ with Lagrange's remainder upto 2 terms:-

Sol:- $f(x) = x^3 - 3x^2 + 2x$, $f(0) = 0$

$f'(x) = 3x^2 - 6x + 2$, are continuous on $[0, 1/2]$

and exist in $(0, 1/2)$

And $f'(0) = 2$, $f''(0) = -6$

$f''(x) = 6x - 6$

Here $a = 0$, $a+h = 1/2$

Taylor's theorem with Lagrange form of remainder

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{\lfloor 2} f''(a) + \frac{h^3}{\lfloor 3} f'''(a) + \cdots$$

$$\cdots + \frac{h^{n-1}}{\lfloor n-1} f^{n-1}(a) + \frac{h^n}{\lfloor n} f^n(a+\theta h) \quad -(1)$$

Here $n = p = 2$

$$f(a+h) = f(1/2) = (1/2)^3 - 3(1/2)^2 + 2 \cdot 1/2$$

$$= \frac{1}{8} - \frac{3}{4} + 1 = \frac{1-6+8}{8} = 3/8$$

$f(0) = 0$, $f'(0) = 2$, $f''(\theta) = 6\theta - 6$

Sub these values in (1), upto $n = 2$

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{\lfloor 2} f''(\theta)$$

Here $h = b - a = 1/2$, $a = 0$, $a+h = 1/2$

$$f(1/2) = f(0) + h f'(0) + \frac{h^2}{\lfloor 2} f''(\tfrac{\theta}{2})$$

$$3/8 = 0 + \frac{1}{2} \cdot 2 + \frac{1}{4 \times 2} \cdot \left(\frac{6\theta}{2} - 6\right)$$

$$3/8 = 1 + \frac{3}{8}(\theta - 2)$$

$\Rightarrow \frac{3}{8} - 1 = \frac{3}{8}(\theta - 2) \Rightarrow -\frac{5}{8} \times \frac{4}{3} = \theta - 1$

$\Rightarrow -5/8 = \frac{3}{8}(\theta - 2) \Rightarrow -\frac{5}{3} = \theta - 2$

$\Rightarrow 2 - \frac{5}{3} = \theta \Rightarrow \theta = \frac{1}{3}$

$\Rightarrow \theta = 1/3 \in (0, 1/2)$

2. Find the Taylor's Series expansion of $\sin 2x$ about $x = \pi/4$ or in powers of $x - \pi/4$ upto 4 terms

Sol:- let $x - \pi/4 = t$

$x = \pi/4 + t$

$\sin 2x = \sin 2(\pi/4 + t) = \sin(\pi/2 + t) = \cos 2t$

$= 1 - \frac{(2t)^2}{2} + \frac{(2t)^4}{4!} + \cdots$

$= 1 - \frac{4t^2}{2} + \frac{16t^4}{24} - \cdots$

$= 1 - \frac{4}{2}(x - \pi/4)^2 + \frac{16}{24}(x - \pi/4)^4 - \cdots$

(3) Show that $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots$

Sol:- Maclaurin's series is given by

$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \cdots$

$f(x) = \sin x$ $f(0) = \sin 0 = 0$

$f'(x) = \cos x$ $f'(0) = \cos 0 = 1$

$f''(x) = -\sin x$ $f''(0) = 0$

$f'''(x) = -\cos x$ $f'''(0) = -1$

$f^{iv}(x) = \sin x$ $f^{iv}(0) = 0$

$f^{v}(x) = \cos x$ $f^{v}(0) = 1$

Substituting these values in (1)

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \dots$$

4) Verify Taylor's theorem for $f(x) = (1-x)^{5/2}$ with Lagrange's form of remainder upto 2 terms in the interval [0 1]

Sol:- Consider $f(x) = (1-x)^{5/2}$ in [0 1]

i) $f(x)$, $f'(x)$ are continuous in [0 1]

(ii) $f''(x)$ is differentiable in (0,1)

Thus $f(x)$ satisfy the conditions of Taylor's theorem

We consider Taylor's theorem with Lagrange's form of remainder

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(\theta x) \quad \text{with } 0<\theta<1 \quad -①$$

Here $n = p = 2$, $a = 0$ and $x = 1$

$f(x) = (1-x)^{5/2} \Rightarrow f(0) = 1$

$f'(x) = -5/2 (1-x)^{3/2} \Rightarrow f'(0) = -5/2$

$f''(x) = \frac{15}{4}(1-x)^{1/2} \Rightarrow f''(\theta x) = \frac{15}{4}(1-\theta x)^{1/2}$

$\Rightarrow f''(\theta) = \frac{15}{4}(1-\theta)^{1/2}$

and $f(1) = 0$

from (1), we have $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(\theta x)$

Substituting the above values we get

$$0 = 1 + 1.\left(\frac{-5}{2}\right) + \frac{1^2}{2!} \cdot \frac{15}{4}(1-\theta)^{1/2}$$

$$\Rightarrow \theta = 9/25 = 0.36 \in (0.1)$$

Thus Taylor's theorem verified

Assignment problems:-

1. Obtain the Maclaurin's Series expansion of the following function
(i) e^x (ii) $\cos x$ (iii) $\sin x$ (iv) $\cosh x$ (v) $\sinh x$

2. Obtain the Maclaurin's series expansion of $\log(1+x)$

3. $f(x) = (1+x)^n$

(4) Obtain Taylor's series expansion of $\sin x$ in powers of $x-\pi/4$

(5) Find the Taylor's series expansion of $\sin 2x$ in " of $x-\pi/2$

(6) Verify Taylor's theorem for $f(x) = (1-x)^{5/2}$ with Lagrange's
2003
form of remainder upto 2 terms in the interval $[0,1]$

(7) Find Maclaurin's theorem with Lagrange form of Remainder
2003
for $f(x) = \cos x$

(8) S.T $\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \cdots$ for $0 < x < 2$

Note: we have to expand $\sqrt{x}$ in powers of $x-1$

(9) Expand $\cos x$ in powers of $(x-\pi/4)$ upto 4 terms

(10) Find the first four terms in the expansion of
$\log(\sin(x+h))$ in ascending powers of h

(11) Expand $e^{x\sin x}$ in powers of x

(12) Expand $(1+x)^n$ in powers of x

(13) Show that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}e^{\theta x}$

(14) Calculate the approximate value of $\sqrt{10}$ to four decimal places using Taylor's theorem $a=9, h=1$

(15) Expand $\log_e x$ in power of $x-1$ & $\log\cos(x+h)$ By Taylor's

(16) Expand $\tan^{-1}x$ in power of x by Maclaurin's theorem

OM SRI SHIRIDI SAI RAM

Assignment Problems for Unit - II

1. Define Continuous function and state the Rolle's theorem.

2. Using Rolle's theorem, show that $g(x) = 8x^3 - 6x^2 - 2x + 1$ has a zero between 0 and 1

3. (a) Verify Rolle's theorem for $f(x) = e^x(\sin x - \cos x)$ in $(\frac{\pi}{4}, \frac{5\pi}{4})$

(b) Apply Rolle's theorem for $\sin x\sqrt{\cos 2x}$ in $[0, \frac{\pi}{4}]$ and find x such that $0 < x < \pi/4$

(c) By considering the function $(x-2)\log x$, show that the equation $x\log x = 2 - x$ is satisfied by atleast one value of x lying between 1 & 2

4. State Lagrange's Mean value theorem and verify Lagrange's mean value theorem for $f(x) = x(x-2)(x-3)$ in $(0, 4)$

5. If $a < b$, prove that $\frac{b-a}{(1+b^2)} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$ using Lagrange's mean value theorem and hence deduce that $\frac{5\pi+4}{20} < \tan^{-1}2 < \frac{\pi+2}{4}$.

6. Prove that (a) $\frac{\pi}{6} + \frac{1}{5\sqrt{3}} < \sin^{-1}\frac{3}{5} < \frac{\pi}{6} + \frac{1}{8}$ (b) $\frac{\pi}{3} + \frac{1}{5\sqrt{3}} > \cos^{-1}\frac{3}{5} > \frac{\pi}{3} - \frac{1}{8}$

using Lagrange's Mean value theorem

7. State Cauchy's Mean value theorem and verify Cauchy's Mean value theorem for $f(x) = \sqrt{x}$ & $g(x) = \frac{1}{\sqrt{x}}$ in $[a, b]$ where $0 < a < b$

8. Define Jacobian and Functional dependence. If $u = x^2 - y^2$, $v = 2xy$ where $x = r\cos\theta$, $y = r\sin\theta$ then show that $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$

Assignment Problems for III UNIT: Beta, Gamma function and Multiple Integrals

1. (a) Define Beta and Gamma functions

(b) Show that $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$

2. (a) Prove that $\frac{\beta(p,q+1)}{q} = \frac{\beta(p+1,q)}{p} = \frac{\beta(p,q)}{p+q}$ where $p>0, q>0$

(b) Show that $\int_a^b (x-a)^m (b-x)^n dx = (b-a)^{m+n+1} \beta(m+1,n+1)$

3. (a) show that $\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}$ where $m>0, n>0$

(or) Find the relation between β & Γ functions

(b) Prove that $\Gamma n \, \Gamma 1-n = \frac{\pi}{\sin \pi}$

4. (a) Show that $\Gamma \frac{1}{2} = \sqrt{\pi}$ (b) show that $\Gamma n = \int_0^1 (\log \frac{1}{x})^{n-1} dx$, $n>0$

5. (a) Evaluate $\int_0^2 x(8-x^3)^{1/3} dx$ (b) show that $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{1}{2}\Gamma\frac{1}{4}\Gamma\frac{3}{4}$

6. (a) Evaluate $\int_0^1 x^4 (\log 1/x)^3 dx$ (b) Evaluate $\int_0^\infty x^{-3/2}(1-e^{-x})dx$

7. Show that $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$ and hence deduce that $\int_0^{\pi/2} \sin^n\theta \, d\theta = \int_0^{\pi/2} \cos^n\theta \, d\theta = \frac{\Gamma\frac{n+1}{2}}{2\Gamma\frac{n+2}{2}} \cdot \sqrt{\pi}$

8. (a) prove that $\int_0^{\pi/2} \sin^2\theta \cos^4\theta \, d\theta = \pi/32$ (b) Prove that $\int_0^\infty e^{-y^{1/m}} dy = m\Gamma m$

9 (a) P.T $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi$ (b) Show that $\int_0^\infty e^{-x^2} \cdot x^4 dx = \frac{3\sqrt{\pi}}{8}$

10. Evaluate $\int_0^\infty \frac{x\,dx}{1+x^6}$ using $\beta-\Gamma$ functions

11. (a) Prove that $\int_0^1 x^m (\log x)^n \, dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$ where n is the integer.

(b) Evaluate $4\int_0^\infty \frac{x^2}{1+x^4}\,dx$ using β-Γ functions

(c) Evaluate $\int_0^a x^4 \sqrt{a^2-x^2}\,dx$

12. (a) Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$ (b) $\int_0^5 \int_0^{x^2} x(x^2+y^2)\,dx\,dy$

13. (a) Evaluate $\iint_R y^2\,dx\,dy$ where R is the region bounded by the parabolas $y^2=4x$ and $x^2=4y$

(b) Find the value of $\iint xy\,dx\,dy$ taken over the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

14. (a) Evaluate $\int_0^{\pi/2}\int_0^\infty \frac{r\,dr\,d\theta}{(r^2+a^2)^2}$ (b)

(b) Evaluate $\iint r^3\,dr\,d\theta$ over the area included between the circles $r=2\sin\theta$ and $r=4\sin\theta$

15. (a) Evaluate the double integral $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dy\,dx$

(b) Change of order of integration $\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$ and hence evaluate the double integral

UNIT · II Functions of Single Variables

Assignment Problems:

1. State Rolle's theorem and verify Rolle's theorem for the following functions

(a) $f(x) = \frac{\sin x}{e^x}$ in $[0, \pi]$ (b) $f(x) = \tan x$ in $[0, \pi]$

(c) $f(x) = 2x^3 + x^2 - 4x - 2$ in $[-\sqrt{3}, \sqrt{3}]$ (d) $g(x) = x^3 - 2x - 3$ in $(1, -3)$

2. Using Rolle's theorem, show that $g(x) = 8x^3 - 6x^2 - 2x + 1$ has a zero between 0 and 1

3. State Lagrange's Mean value theorem and verify Lagrange's mean value theorem for the following functions

(i) $f(x) = x^3 - x^2 - 5x + 3$ in $[0, 4]$ (ii) $f(x) = \log_e x$ in $[1, e]$

4. Using Lagrange's Mean value theorem prove that the following

(i) $\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}\frac{3}{5} > \frac{\pi}{3} - \frac{1}{8}$ (ii) $\frac{\pi}{6} + \frac{1}{5\sqrt{3}} < \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{6} + \frac{1}{8}$

(iii) $1 + x < e^x < 1 + xe^x$ for any x

(iv) $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$ and hence $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

5. ~~Using~~ State Cauchy's Mean value theorem and verify C.M.V. Theorem for the following functions

(i) $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$ in $[a, b]$ (ii) $f(x) = e^x$ & $g(x) = e^{-x}$ in $(3, 7)$

(iii) $f(x) = \log x$, $g(x) = x^2$ in $[a, b]$ with $b > a > 1$.

Now,
the Augmented matrix of A is

$$\tilde{A} = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & 3 & -2 & 2 \\ 5 & 1 & 2 & 13 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 5R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -10 \\ 0 & -4 & -3 & -17 \end{bmatrix} \quad R_3 \to R_3 + 4R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -10 \\ 0 & -4 & -3 & -17 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & -19 & -57 \end{bmatrix}$$

$\therefore \rho(A) = 3$ & $\rho(\tilde{A}) = 3$ = no. of. variables.

$\therefore$ the system is consistent & it has unique solution.

from above Augmented matrix:

$x + y + z = 6$ — ①
$y - 4z = -10$ — ②
$-19z = -57$ — ③

$\therefore$ from ③
$-19z = -57$
$\Rightarrow z = 3$.

Sub $z = 3$ in ② we get
$y - 12 = -10 \Rightarrow y = 2$

Sub z & y values in ① we get x.
$x + 2 + 3 = 6 \Rightarrow x = 1$.

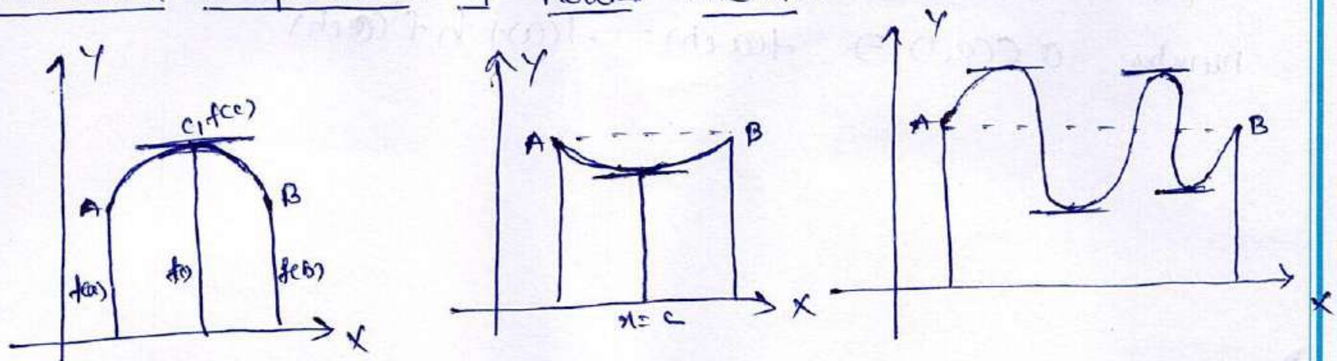
$\therefore$ the solution of system are
$x = 1$; $y = 2$; $z = 3$.

3. find all the non-trivial solutions of
$2x - y + 3z = 0$; $3x + 2y + z = 0$; $x - 4y + 5z = 0$.

sol: Given Equations can be written in matrix form as-

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Geometrical Interpretation of Rolle's theorem:



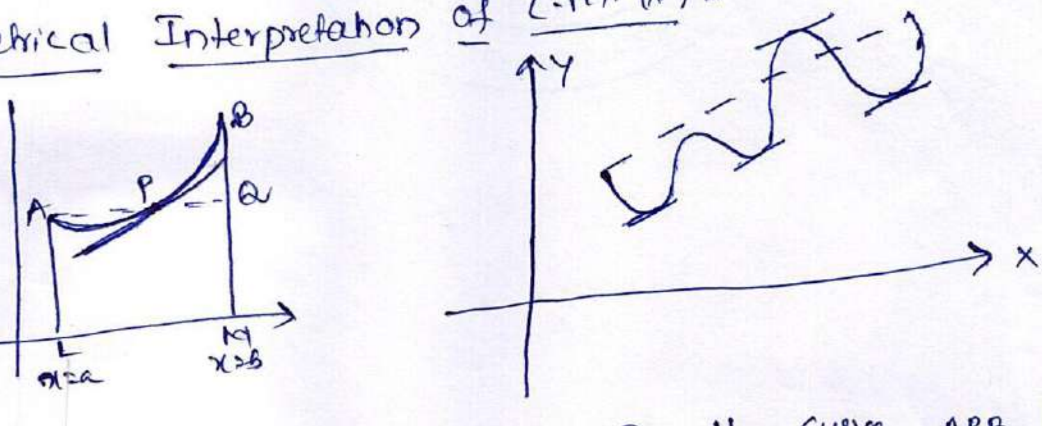
If a function f(x) is continuous function in a closed interval $a \le x \le b$ where $f(a) = f(b)$, then there exists atleast a point b/w a & b. i.e., in the open interval $a < x < b$, say $x = c$ so that the tangent is parallel to x-axis at the point $x = c$

Alternative form of Rolle's theorem:

let $b = a + h$ the $c = a + \theta h$, $0 < \theta < 1$

then $f'(c) = f'(a + \theta h) = 0$ with $0 < \theta < 1$

Geometrical Interpretation of L.M.V.T:-



f(x) Represents the curve APB. If the curve APB has a tangent at all points b/w A & B, the there exists a point $P(x = c)$ b/w A & B such that the tangent at P is parallel to the chord AB.

The slope of the chord $AB = \frac{BQ}{AQ} = \frac{f(b) - f(a)}{b - a}$

The slope of the tangent at $P = f'(c)$. Since the chord at P is parall to the tangent at P. $\therefore f'(c) = \frac{f(b) - f(a)}{b - a}$

Slope of the tangent to the curve at the point 'c' is equal to slope of the chord AB

Another form of L.M.V.T:

If $f(x)$ is differentiable $(a, a+h)$, $\exists$ atleast one number $\theta \in (0,1) \ni f(a+h) = f(a) + h f'(a+\theta h)$

→ If $u = \frac{yz}{x}, v = \frac{zx}{y}, w = \frac{xy}{z}$ then find $J\left(\frac{u,v,w}{x,y,z}\right)$

→ $u = x^2 - 2y$; $v = x+y+z$; $w = x - 2y + 3z$ then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = ?$

→ If $x+y+z = u$, $y+z = uv$, $z = uvw$ then S.T $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$

→ If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ then show that

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta \quad \& \quad \text{find} \quad \frac{\partial(r,\theta,\phi)}{\partial(x,y,z)}$$

→ If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ & $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$$

→ If $u = x^2 - y^2$, $v = 2xy$ where $x = r\cos\theta$, $y = r\sin\theta$ then show that $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$

→ If $x = u(1-v)$, $y = uv$ prove that $JJ' = 1$

$u = x+y$ & $v = \frac{y}{x+y}$ $\quad v_x = \frac{-y}{(x+y)^2}$, $v_y = \frac{x}{(x+y)^2} = \frac{vu' - uv'}{v^2}$

$J = u$ & $J' = \frac{1}{u}$

→ If $x = uv$, $y = u/v$ then verify $\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1$

$\Rightarrow \frac{1}{v} = y/u$

$u = \frac{x}{v} = x\cdot(y/u)$

$\Rightarrow u^2 = xy \Rightarrow u = \sqrt{xy} \Rightarrow v = \frac{u}{y} = \sqrt{x/y}$

$u_x = \sqrt{y}\cdot\frac{1}{2\sqrt{x}}$, $u_y = \frac{\sqrt{x}}{2\sqrt{y}}$

$v_x = \frac{1}{2\sqrt{xy}}$ $\quad v_y = -\frac{\sqrt{x}}{2y^{3/2}}$

→ $(3v - u^2)u = 2w$

→ If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$ then find $J\left(\frac{u,v,w}{x,y,z}\right)$

→ $u = x^2 - 2y$; $v = x+y+z$; $w = x - 2y + 3z$ then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = ?$

→ If $x+y+z = u$, $y+z = uv$, $z = uvw$ then S.T $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$

→ If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ then show that

$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$ & find $\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)}$

→ If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ & $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$

$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$

→ If $u = x^2 - y^2$, $v = 2xy$ where $x = r\cos\theta$, $y = r\sin\theta$ then

Show that $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$

→ If $x = u(1-v)$, $y = uv$ prove that $JJ' = 1$

$u = x+y$ & $v = \frac{y}{x+y}$ $v_x = -\frac{y}{(x+y)^2}$, $v_y = \frac{x}{(x+y)^2} = \frac{vu' - uv'}{v^2}$

$J = u$ & $J' = \frac{1}{u}$

→ If $x = uv$, $y = u/v$ then verify $\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1$

$\Rightarrow \frac{1}{v} = y/u$

$u = \frac{x}{v} = x \cdot (y/u)$

$\Rightarrow u^2 = xy \Rightarrow u = \sqrt{xy} \Rightarrow v = \frac{u}{y} = \sqrt{x/y}$

$u_x = \sqrt{y} \cdot \frac{1}{2\sqrt{x}}$, $u_y = \frac{\sqrt{x}}{2\sqrt{y}}$

$v_x = \frac{1}{2\sqrt{xy}}$ $v_y = -\frac{\sqrt{x}}{2y^{3/2}}$

→ $(3v - u^2)u = 2w$

Assignment - IV

Name - G. Ranjith Kumar
Br: ECE - A, Roll No: 10831A0430

1. Find the rank of the matrix $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & 1 \end{bmatrix}$ by reducing it into the normal form.

Sol: Given matrix $A = \begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & 1 \end{bmatrix}$

$\sim$ $R_2 \to 2R_2 - 3R_1$, $R_3 \to 2R_3 - R_1$

$$\sim \begin{bmatrix} 2 & 3 & 7 \\ 0 & -13 & -6 \\ 0 & -9 & -9 \end{bmatrix} \quad C_2 \to 2C_2 - 3C_1, \; C_3 \to 2C_3 - 7C_1$$

$$\sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & -26 & -12 \\ 0 & -18 & -18 \end{bmatrix} \quad R_3 \to 26R_3 - 18R_2$$

$$\sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & -26 & -12 \\ 0 & 0 & -252 \end{bmatrix} \quad C_2 \to 12C_2 - 26C_3$$

$$\sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & -26 & 0 \\ 0 & 0 & 6552 \end{bmatrix} \quad R_1 \to R_1/2, \; R_2 \to -R_2/26, \; R_3 \to R_3/6552$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [I_3]_{3\times3}$$

$\therefore$ The rank = no. of non-zero rows = 3.

$\therefore$ the rank of matrix A is $R(A) = 3$.

2. Solve the system of linear equations.

$x + y + z = 6$; $2x + 3y - 2z = 2$; $5x + y + 2z = 13$;

Sol: Given system of linear equations can be written in matrix form as.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -2 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 13 \end{bmatrix}$$

16. Find the maximum and minimum of the function

$f(x) = 2(x^2 - y^2) - x^4 + y^4$

17. (a) Examine the extrema of $f(x,y) = x^3 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$

(b) Find the stationary points of $u(x,y) = \sin x \sin y \sin(x+y)$

18. Find the maximum and minimum values of $x+y+z$ subject to $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

✓

19. Find the maximum value of $u = x^2 y^3 z^4$ if $2x + 3y + 4z = a$

20. Divide 24 into three points such that the continued of the first, square of the second and cube of the third is maximum.

$$P^{-1} = \begin{bmatrix} -2 & 5 & -1 \\ -2 & 0 & 4 \\ 28 & -20 & -26 \end{bmatrix}$$

$$AP = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 8 & 15 & 2 \\ 6 & 8 & 1 \\ 4 & 10 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 64-48-8 & 120-64-20 & 16-8-2 \\ 32-18-8 & 60-24-20 & 8-3-2 \\ 24-24+0 & 45-32+10 & 6-4+1 \end{bmatrix}$$

$$AP = \begin{bmatrix} 8 & 36 & 6 \\ 6 & 16 & 3 \\ 4 & 23 & 3 \end{bmatrix}$$

Functional Dependence:-

Let $u = f(x,y)$, $v = \phi(x,y)$ be two given differentiable functions of the two independent variables x and y. (suppose these functions u and v are connected by a relation $F(u,v)=0$, where F is differentiable. Then these functions u and v are said to be functionally dependent on one another (i.e., one function say u is a function of the second function v) if the partial derivatives u_x, u_y, v_x & v_y are not all zero simultaneously.

i.e The functions u and v are said to be functionally dependent if the Jacobian of u and v is zero.

$$\text{i.e } \frac{\partial(u,v)}{\partial(x,y)} = 0$$

Note 1:- If u, v, w are functions of x, y, z then u, v, w are functionally dependent provided

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

Note 2:- In generally, the functional dependence can also be provided for n-functions

Property - I :-

Let u and v are functions of x and y

(Imp) and $J = \frac{\partial(u,v)}{\partial(x,y)}$, $J' = \frac{\partial(x,y)}{\partial(u,v)}$ then $JJ' = 1$

proof:- let $u = f(x,y)$

$$\frac{\partial u}{\partial u} = 1 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} \quad -(1)$$

$$\frac{\partial u}{\partial v} = 0 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} \quad -(2)$$

let $v = \phi(x,y)$

$$\frac{\partial v}{\partial v} = 1 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v} \quad -(3)$$

$$\frac{\partial v}{\partial u} = 0 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} \quad -(4)$$

Now. $$JJ' = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} =$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |I| = 1$$

Property 2:- If u and v are functions of r and s & r, s are functions of x, y. Then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$

Proof:- R.H.S $= \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$

$$= \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} \\ \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)} = \text{L.H.S}$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$$

= x =

Property 3:- Let u, v, w be functions of three independent variables x, y, z. If the functions u, v & w are not independent then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

No Need

Proof:- Given u, v, w are not independent

$\therefore f(u,v,w) = 0$

Differentiating w.r.to x, y, z we get

$$\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x}=0 \quad -(1)$$

$$\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}+\frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial y}=0 \quad -(2)$$

$$\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z}+\frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial z}=0 \quad -(3)$$

eliminating $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial w}$ from (1), (2) and (3) we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{vmatrix} = 0$$

Interchanging rows and columns

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = 0$$

$$\therefore \frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$$

Problems for Jacobian and functional dependent:-

1. $u = 3x+2y-z$, $v = x-y+z$ and $w = x+2y-z$

find Jacobian of $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

$u_x = \frac{\partial u}{\partial x}$

Sol:- $u_x = 3 \quad u_y = 2 \quad u_z = -1 \quad u_y = \frac{\partial u}{\partial y}$

$v_x = 1 \quad v_y = -1 \quad v_z = 1 \quad u_z = \frac{\partial u}{\partial z}$

$w_x = 1 \quad w_y = 2 \quad w_z = -1$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = 2$$

2. Show that $\frac{\partial(u,v)}{\partial(r\theta)} = 6r^3 \sin 2\theta$ and given $u = x^2-2y^2$

$v = 2x^2-y^2$ and $x = r\cos\theta$, $y = r\sin\theta$

Sol:- We know that $\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \frac{\partial(x,y)}{\partial(r\theta)}$

Since $u = x^2-2y^2 \quad v = 2x^2-y^2$

$u_x = \frac{\partial u}{\partial x} = 2x \qquad u_y = \frac{\partial u}{\partial y} = -4y$

$v_x = 4x \qquad v_y = -2y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & -4y \\ 4x & -2y \end{vmatrix} = \cancel{-16xy} + \cancel{8xy}$$

$$= \cancel{-8xy}$$

$$= -4xy + 16xy = 12xy$$

We know that $x = r\cos\theta$ and $y = r\sin\theta$

$\frac{\partial x}{\partial r} = x_r = \cos\theta$ $\qquad \frac{\partial x}{\partial \theta} = x_\theta = -r\sin\theta$

$\frac{\partial y}{\partial r} = y_r = +\sin\theta$ $\qquad \frac{\partial y}{\partial \theta} = +r\cos\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & +r\cos\theta \end{vmatrix}$$

$$= +r\cos^2\theta + r\sin^2\theta$$

$$= r(\sin^2\theta + \cos^2\theta) = r$$

Now $\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$

$$= 12xy \cdot r = 12r \cdot r\cos\theta \cdot r\sin\theta$$

$$= 12r^3 \sin\theta \cdot \cos\theta$$

$$= \cancel{12}^{6} \cdot r^3 \cdot \frac{\sin 2\theta}{\cancel{2}}$$

$$= 6r^3 \sin 2\theta$$

3. $u = \sin x + \sin y \qquad v = \sin(x+y)$

Sol:- $u_x = \frac{\partial u}{\partial x} = \cos x \qquad u_y = \frac{\partial u}{\partial y} = \cos y$

$v_x = \frac{\partial v}{\partial x} = \cos x+y \qquad v_y = \frac{\partial v}{\partial y} = \cos(x+y)$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \cos x & \cos y \\ \cos(x+y) & \cos(x+y) \end{vmatrix}$$

$$= \cos(x+y)\cos x - \cos(x+y)\cos y$$

$$= \cos(x+y)(\cos x - \cos y)$$

4. $u = x^2+y^2, \; v = y$ and $x = r\cos\theta, \; y = r\sin\theta$

$u_x = \frac{\partial u}{\partial x} = 2x \qquad u_y = \frac{\partial u}{\partial y} = 2y$

$v_x = \frac{\partial v}{\partial x} = 0 \qquad v_y = \frac{\partial v}{\partial y} = 1$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 0 & 1 \end{vmatrix} = 2x$$

And $x_r = \frac{\partial x}{\partial r} = \cos\theta \qquad x_\theta = \frac{\partial x}{\partial \theta} = -r\sin\theta$

$y_r = \frac{\partial y}{\partial r} = \sin\theta \qquad y_\theta = \frac{\partial y}{\partial \theta} = r\cos\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$= r\cos^2\theta + r\sin^2\theta$

$= r(\cos^2\theta + \sin^2\theta)$

$= r$

Now $\frac{\partial(x, v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$

$= 2x \cdot r = 2xr$

Functionally Dependence:-

5. $u = x+y+z$, $v = x^2+y^2+z^2$ $w = x^3+y^3+z^3-3xyz$

Sol:- $u_x = 1$ $u_y = 1$ $u_z = 1$

$v_x = 2x$ $v_y = 2y$ $v_z = 2z$

$w_x = 3x^2-3yz$ $w_y = 3y^2-3xz$ $w_z = 3z^2-3xy$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= 1\begin{vmatrix} 2y & 2z \\ 3y^2-3xz & 3z^2-3xy \end{vmatrix} - 1\begin{vmatrix} 2x & 2z \\ 3x^2-3yz & 3z^2-3xy \end{vmatrix}$$

$$+ 1\begin{vmatrix} 2x & 2y \\ 3x^2-3yz & 3y^2-3xz \end{vmatrix}$$

$= 2y(3z^2-3xy) - 2z(3y^2-3xz) - [2x(3z^2-3xy) - 2z(3x^2-3yz)]$

$+ 2x(3y^2-3xz) - 2y(3x^2-3yz)$

$$\therefore = 6yz^2 - 6xy - 6zy^2 + 6xz^2 - 6xz^2 + 6x^2y + 6x^2z - 6yz^2$$
$$+ 6xy^2 - 6x^2z - 6x^2y + 6y^2z$$

$$= 0$$

$\therefore$ u, v and w are functionally dependent

$$uv = x^3 + xy^2 + xz^2 + yx^2 + y^3 + yz^2 + zx^2 + zy^2 + z^3$$
$$= x^3 + y^3 + z^3 + (xy^2 + xz^2 + yx^2 + yz^2 + zx^2 + zy^2)$$

$$u^2 = (x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$$

$$uu^2 = (x+y+z)(x^2 + y^2 + z^2 + 2xy + 2yz + 2zx)$$

$$= x^3 + xy^2 + xz^2 + 2x^2y + 2xyz + 2zx^2$$
$$+ x^2y + y^3 + yz^2 + 2xy^2 + 2y^2z + 2xyz$$
$$+ x^2z + y^2z + z^3 + 2xyz + 2yz^2 + 2z^2x.$$

$$= x^3 + y^3 + z^3 + 3xz^2 + 3x^2y + 3xy^2 + 3y^2z + 3z^2y + 3x^2z$$
$$+ 6xy$$

$$3uv - u^3 = 3x^3 + 3y^3 + 3z^3 + 3xy^2 + 3xz^2 + 3x^2y + 3yz^2 + 3zx^2 + 3zy^2$$
$$- x^3 - y^3 - z^3 - 3xz^2 - 3x^2y - 3xy^2 - 3y^2z - 3z^2y$$
$$- 3x^2z - 6xyz$$

$$= 2x^3 + 2y^3 + 2z^2 - 6xyz$$

$$(3v - u^2)u = 2w$$

$\therefore$ u, v, w are functionally dependent

6. $u = \sin^{-1}x + \sin^{-1}y$ and $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

Sol:- $u_x = \frac{1}{\sqrt{1-x^2}}$ $u_y = \frac{1}{\sqrt{1-y^2}}$

$v_x = \sqrt{1-y^2} + \frac{2yx}{2\sqrt{1-x^2}}$ $v_y = \frac{x \cdot 2y}{2\sqrt{1-y^2}} + \sqrt{1-x^2}$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} + \frac{xy}{\sqrt{1-x^2}} & \frac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \end{vmatrix}$$

$$= \frac{1}{\sqrt{1-x^2}}\left(\frac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}\right) - \frac{1}{\sqrt{1-y^2}}\left(\sqrt{1-y^2} + \frac{xy}{\sqrt{1-x^2}}\right)$$

$$= \frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}} + 1 - 1 + \frac{xy}{\sqrt{1-y^2}\sqrt{1-x^2}} = 0$$

$\sin^2x + \cos^2x = 1$
$\cos^2x = 1 - \sin^2x$
$\cos x = \sqrt{1-(\sin x)^2}$
$\cos \sin^{-1}x = \sqrt{1-(\sin(\sin^{-1}x))^2}$
$= \sqrt{1-x^2}$

u and v are functionally dependence

$\therefore u = \sin^{-1}x + \sin^{-1}y$

$\sin u = \sin(\sin^{-1}x + \sin^{-1}y)$

$= \sin(\sin^{-1}x)\cos(\sin^{-1}y) + \cos(\sin^{-1}x)\cdot\sin(\sin^{-1}y)$

$= x\cdot\sqrt{1-y^2} + y\sqrt{1-x^2}$

$= v$

$\therefore \sin u = v$

Assignment problems

95, 93, 2006

1. If $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(x,y)}{\partial(r,\theta)}$ & $\frac{\partial(r,\theta)}{\partial(x,y)}$. Also show that $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$

S.chand, V.G.S, Right

S.chand, Right (1996)

2. If $u = \frac{x+y}{1-xy}$ and $\theta = \tan^{-1}x + \tan^{-1}y$ find $\frac{\partial(u,\theta)}{\partial(x,y)}$

S.chand, VGS, Right (2006) 2007, 995

3. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$ and find $\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)}$

S.chand, Right

4. If $u = x^2 - 2y$; $v = x+y+z$, $w = x-2y+3z$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$,

S.chand, V.G.S 2006, 1995, 1993

5. If $u = x+y+z$, $uv = y+z$, $uvw = z$. show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$

S.chand 2006

6. If $x = u\cos v$, $y = u\sin v$, prove that $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$

V.G.S

7. If $y_1 = \frac{x_2x_3}{x_1}$, $y_2 = \frac{x_3x_1}{x_2}$ and $y_3 = \frac{x_1x_2}{x_3}$ S.T $\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = 4$

2006

8. ~~If u = x+y+z~~, ~~...~~ If $u = 2xy$, $v = x^2 - y^2$, $x = r\cos\theta$ and $y = r\sin\theta$ find $\frac{\partial(u,v)}{\partial(r,\theta)}$

9. If $x = u(1-v)$, $y = uv$ prove that $JJ' = 1$

10. (i) $u = x\sin y$, $v = y\sin x$ } find $\frac{\partial(u,v)}{\partial(x,y)}$

(ii) $u = e^x\sin y$, $v = x + \log\sin y$ }

11. Calculate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ if $u = \frac{2yz}{x}$, $v = \frac{3zx}{y}$, $w = \frac{4xy}{z}$

12. Calculate $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J^* = \frac{\partial(x,y)}{\partial(u,v)}$ verify that $J \cdot J^* = 1$

~~(i) u = x sin y, v = y sin x (ii)~~

(i) $u = x + \frac{y^2}{x}$, $v = y^2/x$ (ii) $x = e^u\cos v$, $y = e^u\sin v$

(iii) ~~u =~~

13. Calculate $\frac{\partial(u,v)}{\partial(r,\theta)}$ if $u = 2axy$, $v = a(x^2-y^2)$

where $x = r\cos\theta$, $y = r\sin\theta$

(14). If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ & $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$ calculate $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$

Functional dependence:- Verify the given functions are F.D or not

1. $u = e^x \sin y$, $v = e^x \cos y$

2. $u = \frac{x}{y}$, $v = \frac{x+y}{x-y}$

3. $u = x^2 e^{-y}\cosh z$, $v = x^2 e^{-y}\sinh z$, $w = 3x^4 e^{-2y}$

4. If $f_1 = xy+yz+zx$, $f_2 = x^2+y^2+z^2$ & $f_3 = x+y+z$. Determine whether they are functionally dependent. If so find the relation
chand V.G.S [illegible]

5. Find whether f_1 and f_2 are functionally dependent. If so find the relation between them.
where $f_1 = \sin^{-1}x + \sin^{-1}y$, $f_2 = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

6. Find whether $f_1 = \frac{x+y}{x-y}$, $f_2 = \frac{xy}{(x-y)^2}$ are functionally dependent if so find the relation between them

7. Find whether the following functions are functionally dependent and if so find the relations if $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$
chand, V.G.S

8.

Functions of Several Variables:-

The area of ellipse πab. That means it is depending on two variables a and b.

If $u = f(x, y, z, t)$ then x, y, z, t are known as independent variables (or) arguments and 'u' is known as the dependent variable (or) value of the function.

i.e We know that $y = f(x)$ is a function of a single variable x. x is the independent variable and y is the dependent variable. If there are two or more independent variables for a function, we say that function is a function of several variables

Function of Several Variables:-

If for every x and y a unique value $f(x, y)$ is associated, then f is said to be function of the two independent variables x and y and is denoted by $z = f(x, y)$

The values of x and y for which the function is defined is known as the domain of the function.

Ex:- $z = \sqrt{a^2 - x^2 - y^2}$ domain $x^2 + y^2 \leq a^2$

$\left(z = \sqrt{a^2 - (x^2 + y^2)} \right.$ i.e $z \geq 0$ & $\left. a^2 \geq 0 \right)$

δ- Neighbourhood:-

δ-Neighbourhood of a point (a, b) in the xy- plane is a square which center at (a, b) bounded by the four lines $x = a-\delta$, $x = a+\delta$ and $y = b-\delta$ & $y = b+\delta$

ie $a-\delta \leq x \leq a+\delta$, $b-\delta \leq y \leq b+\delta$

Continuity of a function of two variables:-

A function f(x, y) is said to be continuous at (a, b) if $(x, y) \to (a, b)$. [must be unique and same along any path]

Example:- Suppose $z = f(x, y)$ is a function of two variables. Then z is said to be continuous at $x = a$ and $y = b$, if corresponding to a positive number $\epsilon > 0$, we can find a positive number δ such that

If $f(x, y) = f(a, b) < \epsilon$ for all value of x & y such that $a-\delta < x < a+\delta$, $b-\delta < y < b+\delta$

or $\lim_{x \to a} f(x, y) = f(a, b)$. The limit is $y \to b$ Independent of the number by which x approches a and y approches b

Continuity

A function f(x, y) is said to be continuous at a point (a, b) if

$$\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b)$$

(or)

$$\lim_{(h, k) \to (0, 0)} f(a+h, b+k) = f(a, b)$$

→ If f is not continuous at (a,b), it is said to be discontinuous at (a,b).

Limit of a function of Several Variables:-

A function f(x,y) is said to have a limit 'L' as the point (x,y) approaches (a,b) and is denoted as

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

(or) for a given ε>0 we can find a 'δ' such that $|f(x,y) - L| < \epsilon$ ∀ (x,y) in the δ-nhd

i.e $|x-a| < \delta$ and $|y-b| < \delta$

Important Note:-

The limit of a function f(x,y) of two variables is said to exist only when the same value is obtained for the limit along any path in the xy-plane from (x,y) to (a,b) say along $x \to a$ and $y \to b$ or along $y \to b$ & $x \to a$ etc

limit may or may not exist.

If it exists it is unique

✓ Jacobians:

Jacobian* is a functional determinant (whose elements are functions) which is very useful in transformation of variables, from cartesian to polar, cylindrical and spherical coordinates in multiple integrals.

If $u(x,y)$ and $v(x,y)$ are two functions of two independent variables x and y, then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian of u,v with respect to x and y. and

is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$

Similarly, the Jacobian of u,v,w w.r.to x,y & z

is $J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

*→ Carl Gustav Jacob Jacobi (1804–1851) German Mathematician

-T. Taylor's theorem for two variables

Statement:- Let $f(x,y)$ be a function of two independent variables x and y. If h and k be small increments in x and y, then $f(x+h,y+k) = f(x,y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)$

$$+\frac{1}{2!}\left[h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right] + \frac{1}{3!}\left[h^3\frac{\partial^3 f}{\partial x^3} + 3h^2k\frac{\partial^3 f}{\partial x^2\partial y}\right.$$

$$\left.+3hk^2\frac{\partial^3 f}{\partial x\partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}\right] + \cdots \quad (A)$$

Note:-(1) The above Taylor's expansion (A) can also be written as

$$f(x+h,y+k) = f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(x,y) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f$$

$$+\frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x,y) + \cdots$$

Note(2):- put $x=a$ and $y=b$, Taylor's expansion (A) is given by $f(a+h,b+k) = f(a,b) + \left[h\left(\frac{\partial f}{\partial x}\right)_{(a,b)} + k\left(\frac{\partial f}{\partial y}\right)_{(a,b)}\right]$

$$+\frac{1}{2!}\left[h^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)} + 2hk\left(\frac{\partial^2 f}{\partial x\partial y}\right)_{(a,b)} + k^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)}\right] + \cdots \quad (B)$$

Note (3):- put $a+h=x$, $b+k=y$ in (B), Taylor's expansion is given by

$$f(x,y) = f(a,b) + \left[(x-a)\left(\frac{\partial f}{\partial x}\right)_{(a,b)} + (y-b)\left(\frac{\partial f}{\partial y}\right)_{(a,b)}\right]$$

$$+\frac{1}{2!}\left[(x-a)^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)} + 2(x-a)(y-b)\left(\frac{\partial^2 f}{\partial x\partial y}\right)_{(a,b)} + (y-b)^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)}\right]$$

$$+ \cdots \longrightarrow (C)$$

Taylor's expansion (C) is also known as expansion of $f(x,y)$ in powers of $(x-a)$ and $(y-b)$ or expansion about (a,b) or in the neighbourhood (nhd) of (a,b).

Note (4):- put $a=0, b=0$ in (C), we obtain the Maclaurin's series of $f(x,y)$

$$f(x,y) = f(0,0) + \left[x\left(\frac{\partial f}{\partial x}\right)_{(0,0)} + y\left(\frac{\partial f}{\partial y}\right)_{(0,0)}\right] + \frac{1}{2!}\left[x^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} + 2xy\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,0)} + y^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)}\right] + \cdots$$

Problems:-

1. Expand $f(x,y) = x^3+y^3+xy^2$ in powers of $(x-1)$ and $(y-2)$ using Taylor's series expansion.

Sol:- Taylor's series expansion of $f(x,y)$ in powers of $(x-a)$ and $(y-b)$ is

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial f(a,b)}{\partial x} + (y-b)\frac{\partial f(a,b)}{\partial y}\right] + \frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f(a,b)}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2 f(a,b)}{\partial x \partial y} + (y-b)^2\frac{\partial^2 f(a,b)}{\partial y^2}\right] + \frac{1}{3!}\left[(x-a)^3\frac{\partial^3 f(a,b)}{\partial x^3} + 3(x-a)^2(y-b)\frac{\partial^3 f(a,b)}{\partial x^2 \partial y} + 3(x-a)(y-b)^2\frac{\partial^3 f(a,b)}{\partial x \partial y^3} + (y-b)^3\frac{\partial^3 f(a,b)}{\partial y^3}\right] + \cdots$$

in powers of $(x-1)$ & $(y-2)$ is

$$f(x,y) = f(1,2) + \left[(x-1)\frac{\partial f(1,2)}{\partial x^2} + (y-2)\frac{\partial f(1,2)}{\partial y^2}\right] + \frac{1}{2!}\left[(x-1)^2\frac{\partial^2 f(1,2)}{\partial y^2} + 2(x-1)(y-2)\frac{\partial^2 f(1,2)}{\partial x \partial y} + (y-2)^2\frac{\partial^2 f(1,2)}{\partial y^2}\right] + \frac{1}{3!}\left[(x-1)^3\frac{\partial^3 f(1,2)}{\partial x^3} + 3(x-1)^2(y-2)\frac{\partial^3 f(1,2)}{\partial x^2 \partial y} + 3(x-1)(y-2)^2\frac{\partial^3 f(1,2)}{\partial x \partial y^2} + (y-2)^3\frac{\partial^3 f(1,2)}{\partial y^3}\right] + \cdots \quad \text{—(1)}$$

$f(x,y) = x^3 + y^3 + xy^2 \qquad f(1,2) = 13$

$f_x = \frac{\partial f}{\partial x} = 3x^2 + y^2 \qquad f_x(1,2) = \left(\frac{\partial f}{\partial x}\right)_{1,2} = 3 + 4 = 7$

$f_y = \frac{\partial f}{\partial y} = 3y^2 + 2xy \qquad f_y(1,2) = 3\times4 + 2\cdot1\cdot2 = 16$

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 6x \qquad f_{xx}(1,2) = 6$

$f_{yy} = \frac{\partial^2 f}{\partial y^2} = 6y \qquad f_{yy}(1,2) = \frac{\partial f(1,2)}{\partial y^2} = 12$

$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2y \qquad f_{xy}(1,2) = \frac{\partial^2 f(1,2)}{\partial x \partial y} = 4$

$f_{xxx} = \frac{\partial^3 f}{\partial x^3} = 6 \qquad f_{xxx}(1,2) = \frac{\partial^3 f(1,2)}{\partial x^3} = 6$

$f_{yyy} = \frac{\partial^3 f}{\partial y^3} = 6 \qquad f_{yyy}(1,2) = \frac{\partial^3 f(1,2)}{\partial y^3} = 6$

$f_{xxy} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = 0 \qquad f_{xxy}(1,2) = \frac{\partial^3 f(1,2)}{\partial x^2 \partial y} = 0$

$f_{yyx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y \partial x}\right) = 2 \qquad f_{yyx}(1,2) = \frac{\partial^3 f(1,2)}{\partial y^2 \partial x} = 2$

$= \frac{\partial}{\partial y}(2y) = 2$

Sub these values in eq ① we get

$f(x,y) = 13 + 7(x-1) + 16(y-2) + \frac{1}{2}[6(x-1)^2 + 8(x-1)(y-2) + 12(\ldots$

$+ \frac{1}{6}[6(x-1)^3 + 3(0)(x-1)^2(y-2) + 3\cdot(2)(y-1)(y-2)^2 + 6(y-1)^3]$

$= 13 + 7(x-1) + 16(y-2) + 3(x-1)^2 + 4(x-1)(y-2) + 6(y-2)^2$

$+ (x-1)^3 + (x-1)(y-2)^2 + (y-1)^3$

= x =

2. Expand $e^x \sin y$ in powers of x and y

Sol:-

$$f(x,y) = f(0,0) + \left[x \frac{\partial f(0,0)}{\partial x} + y \frac{\partial f(0,0)}{\partial y}\right] + \frac{1}{2!}\left[x^2 \frac{\partial^2 f(0,0)}{\partial x^2} + 2xy \frac{\partial^2 f(0,0)}{\partial x \partial y} + y^2 \frac{\partial^2 f(0,0)}{\partial y^2}\right] + \frac{1}{3!}\left[x^3 \frac{\partial^3 f(0,0)}{\partial x^3} + 3x^2 y \frac{\partial^3 f(0,0)}{\partial x^2 \partial y} + 3xy^2 \frac{\partial^3 f(0,0)}{\partial x \partial y^3} + y^3 \frac{\partial^3 f(0,0)}{\partial x y^3}\right] + \cdots \rightarrow (1)$$

$f(x,y) = e^x \sin y \qquad f(0,0) = 0$

$f_x(x,y) = \frac{\partial f}{\partial x} = e^x \sin y \qquad f_x(0,0) = \frac{\partial f(0,0)}{\partial x} = 0$

$f_y(x,y) = \frac{\partial f}{\partial y} = e^x \cos y \qquad \frac{\partial f(0,0)}{\partial y} = e^0 \cdot \cos 0 = 1$

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = e^x \sin y \qquad \frac{\partial^2 f(0,0)}{\partial x^2} = 0$

$f_{yy} = \frac{\partial^2 f}{\partial y^2} = -e^x \sin y \qquad \frac{\partial^2 f(0,0)}{\partial y^2} = 0$

$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = e^x \cos y \qquad \frac{\partial^2 f(0,0)}{\partial x \partial y} = 1$

$f_{xxx} = \frac{\partial^3 f}{\partial x^3} = +e^x \sin y \qquad \frac{\partial^3 f(0,0)}{\partial x^3} = 0$

$f_{yyy} = \frac{\partial^3 f}{\partial y^3} = -e^x \cos y \qquad \frac{\partial^3 f(0,0)}{\partial y^3} = -1$

$f_{xxy} = \frac{\partial^3 f}{\partial x^2 \partial y} = e^x \cos y \qquad \frac{\partial^3 f(0,0)}{\partial x^2 \partial y} = 1$

$f_{yyx} = \frac{\partial^3 f}{\partial x \partial y^2} = -e^x \sin y \qquad \frac{\partial^3 f(0,0)}{\partial x \partial y^2} = 0$

Sub these values in eq (1)

$$f(x,y) = 0 + (x \cdot 0 + y \cdot 1) + \frac{1}{2}\left[x^2(0) + 2xy(1) + y^2(0)\right] + \frac{1}{6}\left(x^3(0) + 3x^2 y(1) + 3xy^2(0) + y^3(-1)\right] + \cdots$$

$$= y + xy + \frac{1}{2}x^2 y - y^3$$

A Assignment problems for

1. Expand $e^x \log(1+y)$ in the neighbourhood of the point $(0,0)$
2. Expand $\tan^{-1}(y/x)$ in powers of $(x-1)$ and $(y-1)$ upto Second degree terms.
3. Expand $x^2y+3y-2$ in powers of $(x-1)$ and $(y+2)$ using Taylor theorem
4. Expand $\sin xy$ in powers of $(x-1)$, $(y-\pi/2)$ upto second degree terms
5. Expand y^x at $(1,1)$
6. If $f(x,y)=\tan^{-1}xy$ compute $f(0.9,-1.2)$ approximately
7. Using Taylor's theorem Expand $\sin(x+y)$
8. Expand e^{xy} in the neighbourhood of $(1,1)$
9. Expand $e^x\cos y$ near $(1,\pi/4)$
10. Expand the function $e^x\sin y$ in powers of x & y upto terms of 3rd degree
11. Use Taylor's theorem Expand $f(x,y)=x^2+xy+y^2$ in powers of $(x-1)$ and $(y-2)$
12. Find Taylor's expansion of $f(x,y)=\cot^{-1}xy$ in powers of $(x+0.5)$ and $(y-2)$ upto second degree terms. Hence compute $f(-0.4,2.2)$
13. Expand $f(x,y)=x^3+y^3+xy^2$ in powers of $(x-1)$ and $(y-2)$ using Taylor's series Expansion.

Maxima And Minima of Function of two Variables with and without Constraints:-

Definition:-

Let $Z = f(x,y)$ be a function of two independent variables x and y

The function $f(x,y)$ is said to have a Relative maximum at $x=a$, $y=b$ if for small values of h and k

$$f(a+h, b+k) < f(a,b)$$

h and k are being positive and negative

The function $f(x,y)$ is said to have a Relative minimum at $x=a$, $y=b$ if for small values of h and k

$$f(a+h, b+k) > f(a,b)$$

Extreme Value:-

$f(a,b)$ is said to be an extreme value of f, if it is a maximum or minimum value

(I) The necessary conditions for $f(x,y)$ to have a maximum or minimum at (a,b) are

$$p = f_x(a,b) = \frac{\partial f(a,b)}{\partial x} = 0$$

$$q = f_y(a,b) = \frac{\partial f(a,b)}{\partial y} = 0$$

(II) **Sufficient Condition:-**

Suppose that $f_x(a,b)=0$, $f_y(a,b)=0$

and let $\frac{\partial^2 f(a,b)}{\partial x^2} = r$; $\frac{\partial^2 f(a,b)}{\partial x \partial y} = s$, $\frac{\partial^2 f(a,b)}{\partial y^2} = t$

Then (i) $f(a,b)$ is a maximum value if $rt - s^2 > 0$ and $r < 0$

(ii) $f(a,b)$ is a minimum if $rt-s^2>0$ & $r>0$

(iii) $f(a,b)$ is not an extreme value if $rt-s^2<0$

(iv) If $rt-s^2=0$, then $f(x,y)$ fails to have maximum or minimum value, and it needs further investigation.

Note :- Stationary rule:-

$f(a,b)$ is said to be a stationary rule of $f(x,y)$ if $f_x(a,b)=0$, $f_y(a,b)=0$. Thus every extreme rule is a stationary value but the converse may not be true.

Procedure to find maximum & minimum values of $f(x,y)$:-

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and equate to zero. Solve these equations to get the values $x=a$, $y=b$

2. Calculate $r=\frac{\partial^2 f}{\partial x^2}$, $s=\frac{\partial^2 f}{\partial x \partial y}$, $t=\frac{\partial^2 f}{\partial y^2}$

3. (a) Maximum:- If $rt-s^2>0$ and $r<0$ then f has a maximum value at $x=a$, $y=b$

(b) Minimum:- If $rt-s^2>0$ and $r>0$ then f has a maximum value at $x=a$, $y=b$

(c) Saddle point:- If $rt-s^2<0$ then f has neither maximum nor minimum

(d) Failure Case:- If $rt-s^2=0$ further investigation needed.

Problems:-

1. locate the Stationary Points of $u = x^4 + y^4 - 2x^2 + 4xy$. and determine their nature [2007 Aug]

Sol:- Given that $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$p = f_x = \frac{\partial f}{\partial x} = 4x^3 - 4x + 4y$

$q = f_y = \frac{\partial f}{\partial y} = 4y^3 - 4y + 4x$

$r = f_{xx} = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$

$t = f_{yy} = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$

$s = f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 4$

The condition for f to have maximum is

$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$

$\therefore \frac{\partial f}{\partial x} = 0 \Rightarrow 4x^3 - 4x + 4y = 0$ —(1)

$\frac{\partial f}{\partial y} = 0 \Rightarrow 4y^3 + 4x - 4y = 0$ —(2)

(1) + (2) $\Rightarrow x^3 + y^3 = 0$

$\Rightarrow (x+y)(x^2 - xy + y^2) = 0$

$\Rightarrow x = -y$ Since $x^2 - xy + y^2$ has imaginary roots

sub $y = -x$ in (1)

ie $4x^3 - 4x + 4(-x) = 0$

$\Rightarrow 4x^3 - 8x = 0$

$\Rightarrow x^3 - 2x = 0$

$\Rightarrow x(x^2 - 2) = 0$

$\Rightarrow x = 0$ or $\sqrt{2}$ or $-\sqrt{2}$

The values of y are $0, -\sqrt{2}, \sqrt{2}$

There fore the stationary points are $(0,0)$ $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$.

at $(\sqrt{2}, -\sqrt{2})$

$r = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4 = 12(\sqrt{2})^2 - 4 = 24 - 4 = 20$

$s = \frac{\partial^2 f}{\partial x \partial y} = 4, \quad t = \frac{\partial^2 f}{\partial y^2} = 12(-\sqrt{2})^2 - 4 = 20$

$rt - s^2 = 400 - 16 = 384 > 0, \quad r > 0$

Therefore f has minimum at $(\sqrt{2}, -\sqrt{2})$

$f_{min} = (\sqrt{2})^4 + (-\sqrt{2})^4 - 4 - 8 - 4$

$= 4 + 4 - 4 - 8 - 4 = -8$

At $(-\sqrt{2}, \sqrt{2})$, $r = 20$, $t = 20$, $s = 4$

$rt - s^2 > 0$ and $r > 0$

$\therefore$ f has minimum at $(-\sqrt{2}, \sqrt{2})$ also

At $(0,0)$

$rt - s^2 = 0$

$f(0,0) = 0, \quad f(h,0) = h^2(h^2 - 2)$

$f < 0$. if $h^2 - 2 < 0$

$f(h,0) > 0$ if $h^2 - 2 > 0$ or $h^2 > 2$

Therefore in the neighbourhood of $(0,0)$ f has +ve and -ve values.

Therefore f has a neither maximum nor minimum value at $(0,0)$

(2) A rectangular box open at the top is to have volume of 32 cubic ft. Find the dimensions

Solution:- Suppose x, y, z are the dimensions of the box and V is the volume

$V = xyz$, $z = \frac{V}{xy}$ —(1)

Required material for the construction of box is $S = xy + 2xz + 2yz$ [surface are of Rectangular]

$= xy + 2x\frac{V}{xy} + 2y \cdot \frac{V}{xy}$

$= xy + \frac{2V}{y} + \frac{2V}{x}$ —(2)

To find minimum & maximum for S to have mini & maximum $\frac{\partial S}{\partial x} = 0$ and $\frac{\partial S}{\partial y} = 0$

Since V is given 32 cubic ft

$\therefore S = xy + \frac{64}{y} + \frac{64}{x}$

$\frac{\partial S}{\partial x} = y - \frac{64}{x^2}$ and $\frac{\partial S}{\partial y} = x - \frac{64}{y^2}$

$\frac{\partial S}{\partial x} = 0 \Rightarrow y - \frac{64}{x^2} = 0 \Rightarrow y = \frac{64}{x^2}$ —(3)

$\frac{\partial S}{\partial y} = 0 \Rightarrow x - \frac{64}{y^2} = 0 \Rightarrow x = \frac{64}{y^2}$ —(4)

Sub (3) in (4) $x = \frac{64}{(64)^2} \cdot x^{4}$

$\Rightarrow x^3 = 64 \Rightarrow x = 4$, ie $y = 4$

$\therefore r = \frac{\partial^2 S}{\partial x^2} = \frac{128}{x^3}$, $r_{(4,4)} = \left(\frac{\partial^2 S}{\partial x^2}\right)_{4,4} = \frac{128}{64} = 2$

$s = \frac{\partial^2 S}{\partial x \partial y} = 1$ $s(4,4) = 1$

$t = \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3}$ $t_{4,4} = \left(\frac{\partial^2 S}{\partial y^2}\right)_{4,4} = 2$

Now $rt - s^2 = 4 - 1 = 3 > 0$

$rt - s^2 > 0$ and $r > 0$ at $x = y = 4$

$\therefore$ S is minimum at $(4, 4)$

Therefore the dimensions are $(4, 4, 2)$

3. Find the shortest distance from the origin to the surface $xyz^2 = 2$

Sol: Let d be the distance from the origin $(0,0,0)$ to any point (x, y, z) of the given surface then

$$d = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$$

$$d^2 = x^2 + y^2 + z^2$$

Eliminating z^2 using the equation of the surface $xyz^2 = 2$. So replace $z^2 = \frac{2}{xy}$

$$\therefore d^2 = x^2 + y^2 + \frac{2}{xy} = f(x, y)$$

$$p = \frac{\partial f}{\partial x} = 2x - \frac{2}{x^2 y}, \quad f_y = q = \frac{\partial f}{\partial y} = 2y - \frac{2}{xy^2}$$

Solving $p = 0$ & $q = 0$, we get

$$p = \frac{\partial f}{\partial x} = 2x - 2/x^2 y = 0 \Rightarrow \frac{x^3 y - 1}{x^2 y} = 0 \Rightarrow x^3 y = 1 \quad -(1)$$

$$q = \frac{\partial f}{\partial y} = 2y - \frac{2}{xy^2} = 0 \Rightarrow \frac{y^3 x - 1}{xy^2} = 0 \Rightarrow xy^3 = 1 \quad -(2)$$

from (1) and (2) $\Rightarrow x^3 y = 1 = xy^3$

$$\Rightarrow x^3 y - yx^3 = 0$$

$$\Rightarrow xy(x^2 - y) = 0$$

Since we know that $x \neq 0$ & $y \neq 0$

So $x^2 - y^2 = 0 \Rightarrow x^2 = y^2 \Rightarrow x = \pm y = 1$

Thus two stationary points are $(1,1)$ and $(-1,-1)$

$\therefore r = f_{xx} = \frac{\partial^2 f}{\partial x^2} = 2 + \frac{4}{x^3 y}$

$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(2y - \frac{2}{xy^2}\right) = \frac{2}{x^2 y^2}$

$t = \frac{\partial^2 f}{\partial y^2} = 2 + \frac{4}{xy^3}$

At $(1,1)$: $r_0 = f_{xx} = 2 + \frac{4}{1} = 6$

$s = 2, \; t = 6$

$\therefore rt - s^2 = 36 - 4 = 32 > 0, \; r > 0$

At $(-1,-1)$: $r = f_{xx} = 2 + \frac{4}{(-1)(-1)} = 6$

$s = f_{xy} = \frac{2}{1} = 2, \; t = 6$

$\therefore rt - s^2 = 32 > 0$ and $r > 0$ and $xyz^2 = 2$
$\Rightarrow z = \sqrt{2}$

$\therefore$ So minimum occurs at $(1,1,\sqrt{2})$ and $(-1,-1,\sqrt{2})$;

The shortest distance is $\sqrt{1^2 + 1^2 + (\sqrt{2})^2} = \sqrt{4} = 2$

= x =

Assignment problems:-

Chand 1997
1. Find the maximum and minimum values of $x^3 + y^3 - 3axy$

96, 95
2. (Chand) Find the maximum and minimum value of $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

98, 2006
3. (Chand) A rectangular box open at the top is to have volume of 3 cubic ft. Find the dimensions of the box requiring least material for its construction

4. (Chand) Discuss the maxima and minima of $x^2 y + xy^2 - axy$

Chand 2007, Aug 2005
5. Examine the function for extreme values $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

1995
6. (Chand) Find the minimum value of $x^2 + y^2 + z^2$ given that $xyz = a^2$

7. (Chand) Find the volume of the largest parallelopiped that can (1991) be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

8. Show that the function $f(x,y) = x^3 + y^3 - 63(x+y) + 12xy$
Chand is maximum at $(-7,-7)$ and minimum at $(3,3)$

9. Find the extreme values of $u = x^2y^2 - 5x^2 - 8xy - 5y^2$
Chand

10. Discuss the maximum and minimum of $x^2 + y^2 + 6x + 12$

2000S
11. Chand Find the maximum and minimum value of $xy + \frac{a^3}{x} + \frac{a^3}{y}$

2002
12. Chand Examine for minimum and maximum values of $\sin x + \sin y + \sin(x+y)$

13. V.Gs Discuss the maximum or minimum values of u, if $u = ax^3y^2 - x^4y^2 - x^3y^3$

14. V.Gs If $f(x,y,z) = \frac{5xyz}{x+2y+4z}$, find the values of x, y, z for which $f(x,y,z)$ is maximum, subject to the condition $xyz = 8$

2006, 89
15. V.Gs Discuss the maximum and minimum of $u = \sin x \sin y \sin(x+y)$

16. VGS Show that the rectangular parallelopiped of maximum volume that can be inscribed in sphere is a cube

17. V.GS If the perimeter of a triangle is constant, show that the triangle has maximum area when it is an equilateral triangle

18. The sum of three numbers is constant. Prove that their product is a maximum when they are equal.

19. Discuss the maximum and minimum is $x^3y^2(1-x-y)$

2006 Aug
20. Divide a given +ve number into three parts such that their product is maximum & sum is a

2007 Aug
21. Find the points on the surface $z^2 = xy + 1$ nearest to the origin

22. Find the stationary points to the following function
2007 and find the maximum or minimum $u = x^2 + 2xy + 2y^2 + 2x + y$

Laglange's Method of Undetermined Multipliers

In many practical and theoretical problems, it is required to find the maximum or minimum of a function of several variables, where the variables are connected by some relation or condition of 3 independent variables, where x, y, z are related by a known constraint $g(x,y,z)=0$, Then the problem of Constrained extrema consists of finding the extrema of

$$u = f(x,y,z) \quad \text{--- (1)}$$

$$\text{Subject to } g(x,y,z)=0 \quad \text{--- (2)}$$

This problem can be solved by (a) elimination method (b) Implicit differention method (c) Laglange's multipliers method

(a) In elimination method, the constraint (2) is ~~said~~ solved for say one variable z in terms of the other variables x and y, Then z is eliminated from $f(x,y,z)$ resulting in a function of two variables x and y only. The disadvantage of this method is that many times. (2) may not be solvable and in case of solution also the amount of algebra will be generally enormous

(b) In implicit differentiation method, no elimination of variables is done but derivatives are elimina by calculating them through implicit differentiation This method also suffers due to more labour involve

(c) The very useful laglange's method of undetermined multiples introduce an additional unknown constant λ known as laglange's multiplier

Since the stationary values occur when

$f_x = f_y = f_z = 0$, So the total differential

$$df = f_x dx + f_y dy + f_z dz = 0 \quad -(3)$$

Differential of the constraint (2) is

$$dg = g_x dx + g_y dy + g_z dz = 0 \quad -(4)$$

Multiplying (4) by λ and adding to (3), we get

$$(f_x + \lambda g_x) dx + (f_y + \lambda g_y) dy + (f_z + \lambda g_z) dz = 0 - (5)$$

Since x, y, z are independent variables (5) implies that

$$f_x + \lambda g_x = 0 \quad -(6)$$

$$f_y + \lambda g_y = 0 \quad -(7)$$

$$f_z + \lambda g_z = 0 \quad -(8)$$

Solving the four equations (2), (7), (6), (8) for the four unknowns x, y, z, λ we get the required stationary points of $f(x, y, z)$ subject to the constraint (2). Thus the method of Lagrange's multipliers consists of

Step I:- Form the auxiliary equation

$$F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z) \quad -(9)$$

Step II:- Partially differentiate F in (9) w.r.to x, y, z respectively

Step III:- Solve the four equations $F_x = 0$, $F_y = 0$, $F_z = 0$ and the constraint (2) for Lagrange's Multiplier λ and stationary values x, y, z

· OR (Simply)

Sometimes it is required to find the stationary values of a function of several variables which are not all independent but are connected by some given relations. In other words we required to find the extremum of a function subject to some other conditions involving the variables. Such type of problems can be solved by using the method of Lagranges undetermined Multiplier

Working Rule:-

Suppose it is required to find the extremum for the function $f(x,y,z)=0$ subject to the condition $\phi(x,y,z)=0$ — (1)

1. form lagrangean function

$F(x,y,z) = f(x,y,z) + \lambda\, \phi(x,y,z)$ where λ is call the Lagrange multiplier, which is determined by the following conditions.

2. Obtain the equations

$$\frac{\partial F}{\partial x}=0 \quad \text{i.e} \quad \frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0 \qquad — (2)$$

$$\frac{\partial F}{\partial y}=0 \quad \text{i.e} \quad \frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0 \qquad — (3)$$

$$\frac{\partial F}{\partial z}=0 \quad \text{i.e} \quad \frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0 \qquad — (4)$$

3. Solve the equations (1), (2), (3) and (4)

The values of x, y, z so obtained will given the stationary point of $f(x,y,z)$

Note:- To find the maxima or minima for a function $f(x,y,z)=0$ subject to the conditions $\phi(x,y,z)=0$, $\psi(x,y,z)=0$, form the lagrangean function as

$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z) + \mu\psi(x,y,z)$

where λ and μ are Laglange multipliers and proceed as above.

Problems:

1. Find the maximum value of $x^2+y^2+z^2$, given that $xyz = a^3$

Sol:- Let $u = x^2+y^2+z^2$ —(1)

and $\phi = xyz - a^3 = 0$ —(2)

Consider the Laglangean function

$F(x,y,z) = (x^2+y^2+z^2) + \lambda(xyz - a^3)$.

Now $\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial x} + \lambda\cdot\frac{\partial \phi}{\partial x} = 0$

$\Rightarrow 2x + \lambda yz = 0$ —(3)

$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \Rightarrow 2y + \lambda xz = 0$ →(4)

$\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\partial u}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \Rightarrow 2z + \lambda xy = 0$ →(5)

From (3), (4) and (5), we have $\frac{x}{yz} = \frac{y}{zx} = \frac{z}{xy} = -\frac{\lambda}{2}$ →(6)

From first two members, we have $\frac{x}{yz} = \frac{y}{zx} \Rightarrow x^2 = y^2$ →(7)

From last two members, we have $\frac{y}{zx} = \frac{z}{xy} \Rightarrow z^2 = y^2$ →(8)

from (7) and (8), we have $x^2 = y^2 = z^2 \Rightarrow x = y = z$ →(9)

Solving (2) and (9), we get $x^3 - a^3 = 0$

$\Rightarrow x^3 = a^3 \Rightarrow x = a$

$\therefore x = y = z = a$

$\therefore$ Maximum value of $u = a^2 + a^2 + a^2 = 3a^2$

2: Find the minimum value of $x^2+y^2+z^2$ subject to the condition $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$

Sol:- Let $f = x^2+y^2+z^2$ — ①

and $\phi = \frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1=0$ — ②

Consider the Lagrangean function

$F(x,y,z) = f(x,y,z)+\lambda\phi(x,y,z)$

$= (x^2+y^2+z^2)+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)$ — ②

Now $\frac{\partial F}{\partial x}=0 \Rightarrow \frac{\partial f}{\partial x}+\lambda\frac{\partial\phi}{\partial x}=0 \Rightarrow 2x-\frac{\lambda}{x^2}=0$ — ③

$\frac{\partial F}{\partial y}=0 \Rightarrow \frac{\partial f}{\partial y}+\lambda\frac{\partial\phi}{\partial y}=0 \Rightarrow 2y-\frac{\lambda}{y^2}=0$ — ④

$\frac{\partial F}{\partial z}=0 \Rightarrow \frac{\partial f}{\partial z}+\lambda\frac{\partial\phi}{\partial z}=0 \Rightarrow 2z-\frac{\lambda}{z^2}=0$ — ⑤

From ③, ④ and ⑤, we get

$$\left.\begin{aligned} 2x &= \frac{\lambda}{x^2} \Rightarrow 2x^3=\lambda \\ 2y &= \frac{\lambda}{y^2} \Rightarrow 2y^3=\lambda \\ 2z &= \frac{\lambda}{z^2} \Rightarrow 2z^3=\lambda \end{aligned}\right\} \Rightarrow 2x^3=2y^3=2z^3=\lambda$$

$\Rightarrow x^3=y^3=z^3$

$\Rightarrow x=y=z$ — ⑥

Solving ② and ⑥

i.e. $\frac{1}{x}+\frac{1}{x}+\frac{1}{x}=1 \Rightarrow \frac{3}{x}=1 \Rightarrow x=3$

Similarly $x=y=z=3$

$\therefore$ Minimum value $f=3^2+3^2+3^2=9+9+9=27$

---

(3) Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Sol:- Let $2x, 2y, 2z$ be the length, breadth and height of the rectangular parallelopiped that can be inscribed in the ellipsoid

$$\phi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \quad -(1)$$

Then the centroid of the parallelopiped coincides with the center $O(0,0,0)$ of the ellipsoid and the corners of the parallelopiped lie on the surface of the ellipsoid as

If (x,y,z) is any corner of the parallelopiped then it satisfies condition (1)

Let v be the volume of the parallelopiped

i.e $v = 8xyz \quad -(2)$

we have to find the maximum value of v subject to the condition (1)

$$F(x,y,z) = v + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

where λ is the multiplier to determinent such that

$$\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial v}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \Rightarrow 8yz + \frac{2x\lambda}{a^2} = 0 \quad -(3)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial v}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \Rightarrow 8xz + \frac{2y\lambda}{b^2} = 0 \quad -(4)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\partial v}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \Rightarrow 8xy + \frac{2z\lambda}{c^2} = 0 \quad -(5)$$

Now from (3), (4) and (5) are combined as

$$\left.\begin{aligned} 8yz &= \frac{-2x\lambda}{a^2} \Rightarrow \frac{-\lambda}{4} = \frac{a^2yz}{x} \\ 8xz &= \frac{-2y\lambda}{b^2} \Rightarrow \frac{-\lambda}{4} = \frac{b^2xz}{y} \\ 8xy &= \frac{-2z\lambda}{c^2} \Rightarrow \frac{-\lambda}{4} = \frac{c^2xy}{z} \end{aligned}\right\} \frac{-\lambda}{4}.$$

$$-\frac{1}{\lambda} = \frac{a^2yz}{x} = \frac{b^2xz}{y} = \frac{c^2xy}{z} \quad -⑥$$

from second and third functions $\frac{a^2yz}{x} = \frac{b^2xz}{y}$

$$\Rightarrow \frac{y^2}{b^2} = \frac{x^2}{a^2} \quad -⑦$$

Similarly $\frac{y^2}{z^2} = \frac{b^2}{c^2}$ & $\frac{y^2}{b^2} = \frac{z^2}{c^2}$ —⑧

sub ⑦ and ⑧ in ① we have

$$\Rightarrow \frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1$$

$$\Rightarrow \frac{3x^2}{a^2} = 1$$

$$\Rightarrow x^2 = a^2/3$$

$$\Rightarrow x = \frac{a}{\sqrt{3}} \quad \text{lly } y = \frac{b}{\sqrt{3}},\ c = \frac{c}{\sqrt{3}}$$

Hence the possible extreme point is $P\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$

V is maximum at $P\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$ and its maxm value is $V = 8xyz = 8\frac{a}{\sqrt{3}}\cdot\frac{b}{\sqrt{3}}\cdot\frac{c}{\sqrt{3}}$

$$V = \frac{8}{3\sqrt{3}}abc$$

Assignment problems

1. Find the point on the plane which is nearest to the origin $(f = x^2+y^2+z^2)$
2. Find the minimum value of $x^2+y^2+z^2$ if $ax+by+cz=p$
3. Find the maximum and minimum of $u = x^2+y^2+z^2$ if $px+qy+rz$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Interpret the result geometrically
4. Find the maximum and minimum of $x^2+y^2+z^2$ subject to the conditions $ax^2+by^2+cz^2=1$ & $lx+my+nz=0$

5. Find the maxima and minima of $x^2+y^2+z^2$ subject to the conditions. $a_1x+b_1y+c_1z=1$, $a_2x+b_2y+c_2z=1$.

6. Find the minimum value of $x^2+y^2+z^2$, given that $xyz=a^3$

7. Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

8. Find a poin on the plane $3x+2y+z-12=0$ which is nearest to the origin

9. Find the maximum value of $u=x^2y^3z^4$ if $2x+3y+4z=a$

2007
10. Given that $x+y+z=a$. Find that maximum value of $x^my^nz^p$

11. Find the volume of the largest rectangular parallelopiped
2007,2009 that can be inscribed in the ellipsoid of revolution $4x^2+4y^2+9z^2=36$

2007
12. Find the shortest distance from the Surface $xyz^2=2$

15(1)
K9, L5, L9, M6, N0, P3, R9