COURSE HANDOUT

MATHEMATICS-I
(MATRICES AND CALCULUS)

Course Overview:
This course provides mathematical knowledge required to analyze problems encountered in engineering. In this course, the students are acquainted with matrices, solution of system of linear equations, eigen values and eigen vectors, sequence and series, beta and gamma functions, mean value theorems and functions of several variables.

Course Objectives:

1. Determine the rank of the matrix and investigate the solution of system of equations by applying the concepts of consistency.
2. Concepts of Eigen values and Eigen vectors and the nature of quadratic form by finding Eigen values.
3. Concepts of sequence and series and identifying their nature by applying some tests.
4. Mean value theorems geometrical interpretation and their application to the mathematical problems, Evaluation of improper integrals using Beta and Gamma functions.
5. Partial differentiation, Total derivative and finding maxima minima of functions of several variables.

Course Outcomes:

After learning the contents of this course the students must able to:

1. Write the matrix representation of system of linear equations and identify the consistency of the system of equations.
2. Find the Eigen values and Eigen vectors of the matrix and discuss the nature of the quadratic form.
3. Analyse the convergence of sequence and series.
4. Discuss the applications of mean value theorems to the mathematical problems, Evaluation of integrals using Beta and Gamma functions.
5. Examine the extrema of functions of two variables with/without constraints.

Course Syllabus

UNIT-I: Matrices and Linear System of Equations
Matrices and Linear system of equations: Real matrices – Symmetric, Skew – symmetric and Orthogonal. Complex matrices: Hermitian, Skew – Hermitian and Unitary. Rank – Echelon form, Normal form. Solution of linear systems – Gauss Elimination method , Gauss-Jordan method & LU Decomposition method.

UNIT-II: Eigen Values and Eigen Vectors
Eigen values, Eigen vectors – properties, Cayley-Hamilton theorem (without Proof) - Inverse and powers of a matrix by Cayley-Hamilton theorem – Diagonolization of matrix – Quadratic forms: Reduction to canonical form, nature, index and signature.

UNIT-III: Sequences & Series

Basic definitions of Sequences and series, Convergence and divergence, Ratio test, Comparison test, Cauchy's root test, Raabe's test, Integral test ,Absolute and conditional convergence.

UNIT-IV: Beta & Gamma Functions and Mean Value Theorems

Gamma and Beta Functions-Relation between them, their properties – evaluation of improper integrals using Gamma / Beta functions. Rolle's theorem, Lagrange's mean value theorem, Cauchy's mean value theorem, Generalized mean value theorem (all theorems without proof) – Geometrical interpretation of mean value theorems.

UNIT-V: Functions of several variables

Partial differentiation and total differentiation, Functional dependence, Jacobian determinant- Maxima and minima of functions of two variables with constraints and without constraints, Method of Lagrange's multipliers.

Text Books:

1. B.S. Grewal, Higher Engineering Mathematics, Khanna Publishers, 43rd Edition, 2014

2. R.K. Jain, S.R.K. Iyengar, Advanced Engineering Mathematics, Narosa Pulishing, 3rd Edition, 2016

3. B.V. Ramana, Higher Engineering Mathematics, McGraw Hill Education, Chennai, 29th Reprint, 2017

References:

1. G.B.Thomas, R.L. Finney, Calculus and Analytic geometry, 9th Edition, Pearson, 2002

2. Erwin Kreyszig, Advanced Engineering Mathematics, 9th Edition, John Wiley & Sons, 2006

3. Michael Greenberg, Advanced Engineering Mathematics, 2nd Edition, Pearson, 2002

UNIT-I: Matrices and Linear System of Equations

Definitions:

Square matrix: A matrix in which the number rows is equal to the number of columns, is called a square matrix. Thus, $A=[a_{ij}]_{n\times n}$ is a square matrix of order n.

Principal diagonal of a square matrix: Let $A=[a_{ij}]_{n\times n}$ be a square matrix. The elements of a_{ij} of matrix A for which $i=j$ are called the diagonal elements of A. The line along which the diagonal elements lie is called the ***principal diagonal*** of A.

Diagonal matrix: A square matrix in which all non-diagonal elements are zero is called a diagonal matrix. If $d_1, d_2, \ldots, d_n$ are the diagonal elements of a diagonal matrix A, then A is denoted as $A = diag\left(d_1, d_2, \ldots, d_n\right)$.

Example: $D=\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = diag\left(2,-3,5\right)$

Identity matrix: A diagonal matrix in which each diagonal element is unity *i.e.,* 1 is called an identity matrix or a unit matrix. An identity matrix of order n is denoted by I_n.

Example: $I_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I_4=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Upper triangular matrix: A square matrix in which all the elements below the principal diagonal are zero is called an upper triangular matrix.

Example: $U = \begin{bmatrix} 3 & 1 & -2 \\ 0 & 8 & 6 \\ 0 & 0 & -4 \end{bmatrix}$

Lower triangular matrix: A square matrix in which all the elements above the principal diagonal are zero is called a lower triangular matrix.

Example: $L = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 8 & 0 \\ 7 & 9 & 4 \end{bmatrix}$

Triangular matrix: A square matrix is said to be a triangular matrix if it is either upper triangular matrix or lower triangular matrix.

Transpose of a matrix: The matrix obtained from a given matrix A by interchanging its rows and columns is called the transpose of A and is denoted by A^T or A'.

Example: If $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 6 & 9 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 5 & 9 \end{bmatrix}$

Trace of a matrix: The sum of the principal diagonal elements of a square matrix A is called its trace and is denoted by $\text{tr}(A)$.

Example: The trace of the matrix $A = \begin{bmatrix} 5 & 1 & -2 \\ 1 & 7 & 6 \\ -2 & 6 & -4 \end{bmatrix}$ is $\text{tr}(A) = 5+7-4 = 8$

Determinant of matrix: The determinant of a square matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is denoted by $\det(A)$

or $|A|$ and is defined as $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Singular and non-singular matrices: A square matrix A is said to be singular if $|A| = 0$. If $|A| \neq 0$ then A is said to be non-singular.

Note: A^{-1} exists iff $|A| \neq 0$.

Real matrix: A matrix A is said to be real if every element of A is a real number

- A real square matrix A is said to be **Symmetric** if $A^T = A$

Example: $A = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 0 & 6 \\ -2 & 6 & -4 \end{bmatrix}$

- A real square matrix A is said to be **Skew-Symmetric** if $A^T = -A$

Example: $A=\begin{bmatrix} 0 & 3 & 5 \\ -3 & 0 & -2 \\ -5 & 2 & 0 \end{bmatrix}$

Note: The principal diagonal elements of a Skew-symmetric matrix are all zeros.

- A real square matrix A is said to **Orthogonal** if $AA^T = A^T A = I$ or $A^T = A^{-1}$

Example: $A=\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

Properties of Real Matrices:

Property 1: *Every square matrix can be uniquely expressed as the sum of symmetric and skew-symmetric matrices.*

Proof: Let A be any square matrix and $A=P+Q$ where $P=\frac{1}{2}\left(A+A^T\right)$, $Q=\frac{1}{2}\left(A-A^T\right)$

$\therefore P^T=\left[\frac{1}{2}\left(A+A^T\right)\right]^T=\frac{1}{2}\left(A+A^T\right)^T=\frac{1}{2}\left(A^T+(A^T)^T\right)=\frac{1}{2}\left(A+A^T\right)=P \quad [\because (A^T)^T=A]$

$\therefore P$ is a symmetric matrix

Now $Q^T=\left[\frac{1}{2}\left(A-A^T\right)\right]^T=\frac{1}{2}\left(A-A^T\right)^T=\frac{1}{2}\left(A^T-(A^T)^T\right)=\frac{1}{2}\left(A^T-A\right) \quad [\because (A^T)^T=A]$

$=-\frac{1}{2}\left(A-A^T\right)=-Q$

$\therefore Q$ is a skew-symmetric matrix

To prove the sum is unique: If possible, let $A=R+S$ where $R^T=R$ and $S^T=-S$

Now $P=\frac{1}{2}\left(A+A^T\right)=\frac{1}{2}(R+S+(R+S)^T)=\frac{1}{2}(R+S+R^T+S^T)=\frac{1}{2}(R+S+R-S)=R$

Similarly, $Q=\frac{1}{2}\left(A-A^T\right)=\frac{1}{2}(R+S-(R+S)^T)=\frac{1}{2}(R+S-R^T-S^T)=\frac{1}{2}(R+S-R+S)=S$

$\therefore P=R$ and $Q=S$

Thus, every square matrix can be uniquely expressed as the sum of symmetric and skew-symmetric matrices.

Property 2: *The inverse and transpose of an orthogonal matrix are orthogonal.*

Proof: Let A be an orthogonal matrix $\Rightarrow AA^T=A^TA=I \qquad \ldots (1)$

(i) Taking transpose to equation (1), we get $(AA^T)^T=(A^TA)^T=I^T$

$$\Rightarrow (A^T)^T A^T = A^T(A^T)^T = I \qquad [\because I^T=I]$$

$\therefore A^T$ is an orthogonal matrix.

(ii) Taking inverse to equation (1), we get $(AA^T)^{-1}=(A^TA)^{-1}=I^{-1}$

$$\Rightarrow (A^T)^{-1}A^{-1}=A^{-1}(A^T)^{-1}=I \qquad [\because I^{-1}=I]$$

$$\Rightarrow (A^{-1})^T A^{-1}=A^{-1}(A^{-1})^T=I \qquad [\because (A^T)^{-1}=(A^{-1})^T]$$

$\therefore A^{-1}$ is an orthogonal matrix.

Property 3: *If A, B are orthogonal matrices of same order then AB and BA are orthogonal.*

Proof: Let A, B be the orthogonal matrices of same order

$$\Rightarrow AA^T=A^TA=I \text{ and } BB^T=B^TB=I \;..... \;(1)$$

(i) Consider $(AB)(AB)^T=(AB)(B^TA^T)=A(BB^T)A^T=AIA^T=AA^T=I \qquad$ [By (1)]

Now $(AB)^T(AB)=(B^TA^T)(AB)=B^T(A^TA)B=B^TIB=B^TB=I \qquad$ [By (1)]

$$i.e., \; (AB)(AB)^T=(AB)^T(AB)=I$$

$\therefore$ AB is an orthogonal matrix

(ii) Consider $(BA)(BA)^T = (BA)(A^T B^T) = B(A^T A)B^T = BIB^T = BB^T = I$ [By (1)]

Now $(BA)^T(BA) = (A^T B^T)(BA) = A^T(BB^T)A = A^T IA = A^T A = I$ [By (1)]

$i.e.,$ $(BA)(BA)^T = (BA)^T(BA) = I$

$\therefore BA$ is an orthogonal matrix

Property 4: *The determinant of an orthogonal matrix is* ± 1.

Proof: Let A be an orthogonal matrix $\Rightarrow AA^T = A^T A = I$

Consider $AA^T = I$

Applying det on both sides, we get $\det(AA^T) = \det(I)$

$$\Rightarrow \det(A)\det(A^T) = 1 \qquad [\because \det(I) = 1]$$

$$\Rightarrow \det(A)\det(A) = 1 \qquad [\because \det(A^T) = \det(A)]$$

$$\Rightarrow (\det(A))^2 = 1$$

$$\Rightarrow \det(A) = \pm 1$$

Complex matrix: A matrix A is said to be complex if at least one element of A is a complex number.

Example: $A = \begin{bmatrix} 2-3i & 7 \\ 4 & -2i \end{bmatrix}$

Conjugate of a matrix: The matrix obtained by replacing the elements of a complex matrix A by the corresponding conjugate complex numbers is called the conjugate of the matrix A and is denoted by $\overline{A}$.

Transposed conjugate of a matrix: The transposed conjugate of matrix A $i.e.,\left(\overline{A}\right)^T$ and the conjugate of the transpose of matrix A $i.e.,\overline{\left(A^T\right)}$ are equal. Each of them is denoted by A^θ

Thus, $A^\theta = \left(\overline{A}\right)^T = \overline{\left(A^T\right)}$

Example: $A = \begin{bmatrix} 3+i & 1 & 1-i \\ 2 & 2i & 2+3i \\ 0 & -i & -7+6i \end{bmatrix} \Rightarrow \overline{A} = \begin{bmatrix} 3-i & 1 & 1+i \\ 2 & -2i & 2-3i \\ 0 & i & -7-6i \end{bmatrix}$

$$\therefore \quad A^\theta = \left(\overline{A}\right)^T = \begin{bmatrix} 3-i & 2 & 0 \\ 1 & -2i & i \\ 1+i & 2-3i & -7-6i \end{bmatrix}$$

- A complex square matrix A is said to be **Hermitian** if $A^\theta = A$ $i.e.,$ $A^T = \overline{A}$

Example: $A = \begin{bmatrix} 3 & 1 & -2+3i \\ 1 & 0 & 6i \\ -2-3i & -6i & -4 \end{bmatrix} \Rightarrow \overline{A} = \begin{bmatrix} 3 & 1 & -2-3i \\ 1 & 0 & -6i \\ -2+3i & 6i & -4 \end{bmatrix}$

$$\Rightarrow A^\theta = (\overline{A})^T = \begin{bmatrix} 3 & 1 & -2+3i \\ 1 & 0 & 6i \\ -2-3i & -6i & -4 \end{bmatrix} = A$$

$\therefore A$ is a Hermitian matrix

Note: The principal diagonal elements of a Hermitian matrix are all real.

- A complex square matrix A is said to be **Skew-Hermitian** if $A^\theta = -A$ $i.e.,$ $A^T = -\overline{A}$

Example: $A = \begin{bmatrix} i & 2+i & -3+2i \\ -2+i & -2i & 4+3i \\ 3+2i & -4+3i & 0 \end{bmatrix} \Rightarrow \overline{A} = \begin{bmatrix} -i & 2-i & -3-2i \\ -2-i & 2i & 4-3i \\ 3-2i & -4-3i & 0 \end{bmatrix}$

$$\therefore \quad A^{\theta} = (\overline{A})^T = \begin{bmatrix} -i & -2-i & 3-2i \\ 2-i & 2i & -4-3i \\ -3-2i & 4-3i & 0 \end{bmatrix} = -\begin{bmatrix} i & 2+i & -3+2i \\ -2+i & -2i & 4+3i \\ 3+2i & -4+3i & 0 \end{bmatrix} = -A$$

i.e., $A^{\theta} = A \Rightarrow A$ is a skew-Hermitian matrix

Note: The principal diagonal elements of a Skew- Hermitian matrix are either zeros or purely imaginary

- A complex square matrix A is said to be **Unitary** if $AA^{\theta} = A^{\theta}A = I$ or $A^{\theta} = A^{-1}$

Example: $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \Rightarrow A^{\theta} = \left(\overline{A}\right)^T = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

Now $AA^{\theta} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \times \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Similarly, we can prove that $A^{\theta}A = I$

$\therefore A$ is a unitary matrix.

Properties of Complex Matrices:

i) Every square matrix can be uniquely expressed as the sum of Hermitian and skew- Hermitian matrices.

ii) Every Hermitian matrix can be written as $A+iB$ where A is real and symmetric and B is real and skew- symmetric.

iii) Every Skew-Hermitian matrix can be written as $A+iB$ where A is real and skew-symmetric and B is real and symmetric.

iv) The inverse and transpose of a unitary matrix are unitary.

v) The product of two unitary matrices is a unitary matrix.

Elementary transformations on a matrix:

Any one of the following operations on a matrix is called an elementary transformation.

(i) $R_i \leftrightarrow R_j$: Interchange of i^{th} row and j^{th} row.

(ii) $R_i \rightarrow kR_i$: Multiplication of each element of i^{th} row with a non-zero constant k.

(iii) $R_j \rightarrow R_j + kR_i$: Addition of k times the elements of i^{th} row to the corresponding elements of j^{th} row.

The corresponding column transformations are denoted by $C_i \leftrightarrow C_j, C_i \rightarrow kC_i, C_j \rightarrow C_j + kC_i$

Equivalence of matrices: If a $m \times n$ matrix B is obtained from a given $m \times n$ matrix A by finite number of elementary transformations on A, , then A is said to be equivalent to B.

Symbolically, we can write $A \sim B$

Minor of a matrix: Let A be a matrix of order $m \times n$. The determinant of a square sub-matrix of order r of matrix A is called its minor of order r.

Rank of a matrix: A positive integer r is said to be rank of a non-zero matrix A of order $m \times n$ if it has at least one non zero minor of order r and every minor of order $(r+1)$ is zero.

The rank of the matrix A is denoted by $\rho(A)$.

Properties:

i) If A is equivalent to B *i.e.,* $A \sim B$ then $\rho(A) = \rho(B)$

ii) Rank of a matrix A and its transpose are the same *i.e.,* $\rho(A) = \rho(A^T)$

iii) Rank of a null matrix is zero

iv) If A is a non-zero matrix then $\rho(A) \geq 1$

v) If A is a non-singular matrix of order r then $\rho(A) = r$

vi) If A is a square matrix of order n and $\rho(A) < n$ then $|A| = 0$ *i.e.,* A is singular.

vii) If A is a matrix of order $m \times n$, then $\rho(A) \leq \min\{m, n\}$

viii) Rank of the identity matrix I_n is n

Zero row and Non-zero row: If all the elements in a row of a matrix are zeros, then it is called a zero row and if there is at least one non-zero element in a row then it is called a non-zero row.

Row reduced echelon form of a matrix:

A matrix is said to be in echelon form if it satisfies the following conditions

i) Zero rows, if any, must be below the non-zero row

ii) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

Note: i) The number of non-zero rows in row reduced echelon form of a matrix is equal to its rank

ii) Use elementary row operations only to reduce the matrix to echelon form.

iii) Elementary transformations do not alter (effect) the order and rank of a matrix.

Normal form of a matrix: Every non-zero matrix A of rank r can be reduced by a sequence of elementary transformations, to one of the forms I_r, $\begin{bmatrix} I_r & O \end{bmatrix}$, $\begin{bmatrix} I_r \\ O \end{bmatrix}$, $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$, called the normal form of A, where I_r is the identity matrix of order r.

Note: Use elementary row and column transformations to reduce the matrix into normal form.

System of linear equations:

Consider the system of m linear equations in n-unknowns $x_1, x_2, \ldots, x_n$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = b_m$$

The matrix representation of above system is $AX = B$ . . . (1)

where $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{bmatrix}$

Here A is the coefficient matrix, X is the column variable matrix and B is the column constant matrix.

The system $AX = B$ is said to be

(i) **Non-Homogeneous** if $B \neq O$

(ii) **Homogeneous** if $B = O$

The Augmented matrix of the system (1) is denoted by $[A \mid B]$ and defined as

$$[A \mid B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right]$$

Conditions for consistency of non-homogeneous system of linear equations:

Consider the non-homogeneous system $AX = B$

i) If $\rho(A) = \rho(A \mid B) = r = n$ (number of unknowns) then the system $AX = B$ is consistent and has unique solution.

ii) If $\rho(A) = \rho(A \mid B) = r < n$ (number of unknowns) then the system $AX = B$ is consistent and has an infinite number of solutions in terms $(n - r)$ arbitrary constants.

iii) If $\rho(A) \neq \rho(A \mid B)$ then the system $AX = B$ is inconsistent *i.e.* it has no solution at all.

Procedure to find the solution of linear system non-homogeneous equations using rank method:

i) Write the given system in the form $AX = B$

ii) Write the augmented matrix $[A \mid B]$

iii) Reduce the augmented matrix $[A \mid B]$ into echelon form and then solve for the unknowns by back substitution.

Solution of system of homogeneous linear equations:

Consider the homogeneous system $AX = 0$ in n unknowns $x_1, x_2, ..., x_n$, where A is coefficient matrix

i) If $\rho(A) = r = n$ (Number of unknowns) then the system $AX = 0$ has a trivial solution (zero solution).

ii) If $\rho(A) = r < n$ (Number of unknowns) then the system $AX = 0$ has an infinite number of non trivial solutions in terms $(n - r)$ arbitrary constants.

Note: (i) The homogeneous system $AX = 0$ always has a solution

(ii) The homogeneous system $AX = 0$ has a non-trivial solution if $|A| = 0$

Gauss Elimination Method: This method solves a given system of n equations in n unknowns by transforming the coefficient matrix, into an upper triangular matrix and then solve for the unknowns by back substitution.

Consider a system of 3 equations in 3 unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

The above system can be written as $AX = B$ . . . (1)

Consider augmented matrix $[A \mid B] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$

In this method the coefficient matrix A is brought to an upper triangular matrix by elementary row operations. The augmented matrix takes the following form

$$[A \mid B] = \left[\begin{array}{ccc|c} c_{11} & c_{12} & c_{13} & d_1 \\ 0 & c_{22} & c_{23} & d_2 \\ 0 & 0 & c_{33} & d_3 \end{array}\right]$$

Then the solution is obtained by back substitution.

Gauss-Jordan Method: This method is modification of Gauss elimination method. In this method the coefficient matrix A the system of equations $AX = B$ is brought to an identity matrix by elementary row operations.

The augmented matrix takes the following form

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & l_2 \\ 0 & 0 & 1 & l_3 \end{array}\right]$$

Then the solution is obtained without the necessity of back substitution.

***LU*-Decomposition Method:** Consider a non-homogeneous system of 3 equations in 3 unknowns $AX = B \ldots (1)$

A non-singular matrix A is said to have a triangular factorization or LU-Decomposition if A can be expressed as the product of a lower triangular matrix L with ones on its main diagonal and an upper triangular matrix U *i.e.,* $A = LU \ldots (2)$

For $n = 3$, we have $A_{3\times3} = L_{3\times3}U_{3\times3}$

$$i.e., \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

The condition for non-singularity of A implies that $u_{ii} \neq 0$ for all i.

Substituting (2) in (1), we get $LUX = B \ldots (3)$

Put $Y = UX \ldots (4)$ then (3) becomes $LY = B \ldots (5)$

Solve first (5) for Y using forward substitution and then solve (4) for X using backward substitution.

Multiple Choice Questions:

1. If the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & p & 7 \\ 3 & 6 & 10 \end{bmatrix}$ is 2 then $p = \ldots\ldots$

   A) 2 B) 3 C) 4 D) -3

   Answer: A

2. A real matrix $A = \left[a_{ij}\right]_{n\times n}$ is defined as $a_{ij} = i.j \ \forall i, j$ then rank of A is......

   A) $n-1$ B) n C) 1 D) $n-2$

   Answer: C

3. If A is a 3×4 matrix such that the system $AX = B$ is inconsistent then the highest possible rank of A will be.....

   A) 1 B) 2 C) 3 D) 4

   Answer: B

4. If $A = \left[a_{ij}\right]_{20\times20}$ be a matrix such that $a_{ij} = \min\{i, j\};\ i, j = 1, 2, \ldots.20$. Then the rank of $A = \ldots..$

   A) 19 B) 10 C) 20 D) 1

   Answer: C

5. The system of equations $3x - y + 4z = 3,\ x + 2y - 3z = -2, 6x + 5y + \lambda z = -3$ have an infinite number of solutions for value of λ given by

   A) -7 B) 7 C) 5 D) -5

   Answer: D

6. If A and B are non-singular matrices of order n then which of the following statement is not true

A) $\det(AB) = \det(A)\det(B)$ B) $\det\left(A^T\right) = \det(A)$

C) $\det(A+B) = \det(A)+\det(B)$ D) $\det\left(AA^{-1}\right) = 1$

Answer: C

7. If a matrix A is decomposed into its symmetric part $P = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 5 \\ 1 & 5 & 2 \end{bmatrix}$ and skew-symmetric part

$Q = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}$ then $A = ...$

A) $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 4 \\ -1 & 6 & 2 \end{bmatrix}$ B) $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 4 \\ -1 & 4 & 2 \end{bmatrix}$ C) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & 6 & 0 \end{bmatrix}$ D) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 6 & 2 \end{bmatrix}$

Answer: A

8. If $X_{4\times3}, Y_{4\times3}, P_{2\times3}$ are three non-zero matrices then order of the matrix $\left[P\left(X^TY\right)^{-1}P^T\right]^T$ is.....

A) 3×4 B) 4×3 C) 2×2 D) 3×3

Answer: C

9. Suppose $M = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ x & 3 \end{bmatrix}$ is a matrix such that and $M^T = M^{-1}$ then $x = ...$

A) 4 B) -4 C) 5 D) -5

Answer: B

10. In solving system of equations $AX = B$ by Gauss-Jordan method, the coefficient matrix A is reduced to ………… matrix.

A) Identity B) Diagonal C)Upper triangular D) Lower triangular

Answer: A

11. If the system $\begin{bmatrix} k & k & k \\ 0 & k-1 & k-1 \\ 0 & 0 & k^2-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has only one linearly independent solution then $k = ...$

A) $0, 1$ B) $0, -1$ C) $1, -1$ D) $0,\ 1, -1$

Answer: B

12. The determinant of the matrix $P = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 2 & 1 \end{bmatrix}$ is …..

A) 1 B) 2 C) 3 D) 4

Answer: D

Linear Transformation: Consider a set of n linear equations

$$\left.\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ y_n &= a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n \end{aligned}\right\} \quad \dots\dots\dots (1)$$

Let $Y = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$

Then set of n equations (1) can be represented as $Y = AX \dots (2)$, which transforms the set of n variables $(x_1, x_2, .., x_n)$ into the set of n variables $(y_1, y_2, ..., y_n)$. Thus (2) is a transformation which transforms X into Y. Here A is known as the matrix of the transformation.

The transformation $Y = AX$ is said to be

i) **linear** if $A(c_1X_1 + c_2X_2) = c_1(AX_1) + c_2(AX_2) = c_1Y_1 + c_2Y_2$, where c_1, c_2 are constants.

ii) **regular** if A is non singular matrix *i.e.,* $|A| \neq 0$

iii) **orthogonal** if A is orthogonal matrix *i.e.,* $A^{-1} = A^T$

The inverse transformation of $Y = AX$ is given by $X = A^{-1}Y$.

Eigen values and Eigen vectors: Let A be an $n \times n$ matrix. Suppose the linear transformation $Y = AX$ transforms X into a scalar multiple of itself *i.e.,* $AX = Y = \lambda X$ then the scalar λ is known as the eigen value or characteristic root and the corresponding non zero vector X is known as the eigen vector or characteristic vector of A.

Example. Let $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$, $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now $AX_1 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \lambda_1 X_1$

$\therefore \lambda_1 = 1$ is the eigen value of A corresponding to the eigen vector $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Now $AX_2 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 7 \\ 4 \end{bmatrix} \neq \lambda_2 X_2$

$\therefore X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is not an eigen vector of A

Characteristic equation: If λ is an eigen value of A corresponding to the eigen vector X, then

$$\begin{aligned} AX = \lambda X &\Rightarrow AX = \lambda IX \\ &\Rightarrow AX - \lambda IX = 0 \\ &\Rightarrow (A - \lambda I)X = 0 \end{aligned}$$

Thus $|A - \lambda I| = 0$ is known as the characteristic equation of A.

Note: i) The roots of characteristic equation of A are the eigen values of A.

ii) If all the n eigen values of A are distinct, then there correspond n distinct linearly independent eigen vectors

iii) The algebraic multiplicity of an eigen value λ is its order as a root of the characteristic equation (*i.e.,* if λ is repeated m times then its algebraic multiplicity is m)

iv) The geometric multiplicity of λ is the number of linearly independent eigen vectors corresponding to λ.

Procedure to find eigen values and eigen vectors of A :

i) Solve the characteristic equation $|A-\lambda I|=0$ for the eigen values $\lambda_1, \lambda_2, ..., \lambda_n$.

ii) For a specific eigen value λ_i, solve the homogeneous system $(A-\lambda_i I)X = O$, then we get the eigen vector of A corresponding to λ_i

Note: The characteristic equation of $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is $|A-\lambda I|=0$

$$i.e., \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - (\text{tr}(A))\lambda^2 + (\text{Sum of the minors of principal diagonal elements of } A)\lambda - \det(A) = 0$$

$$\Rightarrow \lambda^3 - (a_{11}+a_{22}+a_{33})\lambda^2 + \left[\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right]\lambda - \det(A) = 0$$

Properties of Eigen values and Eigen vectors:

Property 1: Any square matrix A and its transpose A^T have the same eigen values.

Property 2: The eigen values of a triangular matrix are just the diagonal elements of the matrix.

Property 3: The eigen values of a diagonal matrix are its diagonal elements

Property 4: The sum of the eigen values of matrix A is trace of A

Property 5: The product of the eigen values of a matrix A is equal to its determinant.

Property 6: If λ is an eigen value of a matrix A then $\frac{1}{\lambda}$ is an eigen value of A^{-1} .

Property 7: If λ is an eigen value of an orthogonal matrix A then $\frac{1}{\lambda}$ is also its eigen value.

Property 8: If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigen values of the matrix A then A^m has the eigen values $\lambda_1^m, \lambda_2^m, ..., \lambda_n^m$ (m being a positive integer)

Property 9: If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigen values of matrix A then $\lambda_1 \pm k, \lambda_2 \pm k, ..., \lambda_n \pm k$ are the eigen values of $A \pm kI$.

Property 10: If λ is an eigen value of a non singular matrix A, then $\frac{|A|}{\lambda}$ is an eigen value of the matrix $adj\,A$.

Property 11: The eigen values of an orthogonal matrix are of unit modulus.

Property 12: The eigen values of a Hermitian matrix are real.

Property 13: The eigen values of a Skew-Hermitian matrix are either zero or purely imaginary.

Property 14: The eigen values of an unitary matrix have absolute value 1.

Cayley-Hamilton theorem: Every square matrix satisfies its own characteristic equation *i.e.,* if the characteristic equation of a nth order square matrix A is $\lambda^n + k_1\lambda^{n-1} + ... + k_{n-2}\lambda^2 + k_{n-1}\lambda + k_n = 0$ then $A^n + k_1A^{n-1} + ... + k_{n-2}A^2 + k_{n-1}A + k_nI = O$

Similar Matrices: Let A and B be square matrices of same order. The matrix A is said to be similar to the matrix B if there exists a non-singular matrix P such that $A = P^{-1}BP$ or $PA = BP$

Note: If two matrices are similar, then they have the same characteristic equation and hence the same eigen values.

Example: Show that the matrices $A = \begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ are similar to each other

Solution. The given matrices are similar if there exists a non-singular matrix P such that $PA = BP$

Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix such that $PA = BP$

$$i.e., \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5a-2b & 5a \\ 5c-2d & 5c \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ -3a+4c & -3b+4d \end{bmatrix}$$

Equating the corresponding elements, we obtain

$5a - 2b = a + 2c \Rightarrow 4a - 2b - 2c = 0 \ ...(i)$; $5a = b + 2d \Rightarrow 5a - b - 2d = 0 \ ...(ii)$; $5c - 2d = -3a + 4c \Rightarrow 3a + c - 2d = 0 \ ..iii)$; $5c = -3b + 4d \Rightarrow 3b + 5c - 4d = 0 \ ...(iv)$

Solving the above equations, we get $a = 1,\ b = 1,\ c = 1,\ d = 2.$

$\therefore\ P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, which is a non-singular matrix.

Hence the matrices A and B are similar to each other

Diagonalization of a matrix:

A square matrix A is diagonalizable if it is similar to a diagonal matrix *i.e.,* there exists a non-singular matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix. Here P is known as the *modal matrix* and D is known as the *spectral matrix* of A. Since similar matrices have the same eigen values, the diagonal elements of D are the eigen values of A.

Theorem: A square matrix A of order n is diagonalizable if and only if it has n linearly independent eigen vectors.

Note:

1) A square matrix A of order n has always n linearly independent eigen vectors when its eigen values are distinct.
2) For every eigen value λ of a matrix A, the geometric multiplicity $(\lambda) \leq$ algebraic multiplicity (λ).
3) A square matrix A is diagonalizable if and only if the geometric multiplicity is equal to the algebraic multiplicity for every eigen value of A.

Procedure to diagonalization and calculation of powers:

i) Find the eigen values and the corresponding eigen vectors of A.
ii) If the geometric multiplicity is equal to the algebraic multiplicity for every eigen value of A, then form the modal P by taking the eigen vectors as columns.
iii) Calculate P^{-1}.
iv) Find the spectral matrix $D = P^{-1}AP \ ...(1)$
v) Premultiplying (1) by P and post-multiplying (i) by P^{-1}, we get $PDP^{-1} = A \ ... \ (2)$

From (2), we obtain $A^2 = A\cdot A = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$

Similarly, $A^3 = PD^3P^{-1}$, $A^4 = PD^4P^{-1}$,....

In general, $A^n = PD^nP^{-1}$ for any positive integer n.

Note: For any matrix polynomial $Q(A)$, we have $Q(A) = PQ(D)P^{-1}$

Quadratic Form: A homogeneous expression of second degree in $n(\geq 2)$ variables is called a *quadratic form. i.e.,* An expression of the form $Q = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$ ... (1), where $a_{ij} = a_{ji}$ are real, is called a quadratic form in n variables $x_1, x_2, \ldots, x_n$.

- Every quadratic form corresponding to a symmetric matrix A can be expressed in matrix form as $Q = X^TAX$, where A is known as the matrix of the quadratic form and $X = [x_1, x_2, \ldots, x_n]^T$.

Examples: *i)* $2x^2 + 4xy - 9y^2$ is a quadratic form in two variables x, y

ii) $x_1^2 + 2x_2^2 - 13x_3^2 - 2x_1x_2 + 6x_1x_3 + 8x_2x_3$ is a quadratic form in three variables x_1, x_2, x_3.

- The real symmetric matrix A of the QF $X^TAX = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{12}x_1x_2 + a_{23}x_2x_3 + a_{13}x_1x_3$ is given by

$$A = \begin{bmatrix} coeff\left(x_1^2\right) & \frac{1}{2}coeff\left(x_1x_2\right) & \frac{1}{2}coeff\left(x_1x_3\right) \\ \frac{1}{2}coeff\left(x_1x_2\right) & coeff\left(x_2^2\right) & \frac{1}{2}coeff\left(x_2x_3\right) \\ \frac{1}{2}coeff\left(x_1x_3\right) & \frac{1}{2}coeff\left(x_2x_3\right) & coeff\left(x_3^2\right) \end{bmatrix} = \begin{bmatrix} a_{11} & \frac{1}{2}(a_{12}) & \frac{1}{2}(a_{13}) \\ \frac{1}{2}(a_{12}) & a_{22} & \frac{1}{2}(a_{23}) \\ \frac{1}{2}(a_{13}) & \frac{1}{2}(a_{23}) & a_{33} \end{bmatrix}$$

Example: Write down the symmetric matrix of the following quadratic forms:

a) $x^2 - 4xy + 5y^2$ *b)* $x_1^2 + 3x_2^2 - 2x_3^2 + 2x_1x_2 - 6x_1x_3 - 4x_2x_3$

Solution: *a)* Let $X^TAX = x^2 - 4xy + 5y^2$

The matrix of the quadratic form is $A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$

b) Let $X^TAX = x_1^2 + 3x_2^2 - 2x_3^2 + 2x_1x_2 - 6x_1x_3 + 4x_2x_3$

The matrix of the quadratic form is $A = \begin{bmatrix} 1 & 1 & -3 \\ 1 & 3 & 2 \\ -3 & 2 & -2 \end{bmatrix}$

Example: Write down the quadratic form corresponding to the following symmetric matrices:

a) $A = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$ *b)* $A = \begin{bmatrix} 1 & 3 & -5 \\ 3 & 2 & 0 \\ -5 & 0 & -4 \end{bmatrix}$

Solution: *a)* Let $Q = X^TAX$ be the required quadratic form, where $X = [x_1 \quad x_2]^T$

$$\therefore Q = X^TAX = [x_1 \quad x_2]\begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = = x_1^2 - 3x_2^2 + 4x_1x_2$$

b) Let $Q = X^TAX$ be the required quadratic form, where $X = [x_1 \quad x_2 \quad x_3]^T$

$$\therefore Q = X^TAX = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 1 & 3 & -5 \\ 3 & 2 & 0 \\ -5 & 0 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1^2 + 2x_2^2 - 4x_3^2 + 6x_1x_2 - 10x_1x_3$$

Canonical Form: The canonical form or sum of the squares form of a quadratic form $Q = X^T AX$ in n variables $x_1, x_2, \ldots, x_n$ is another quadratic form $Q' = Y^T DY = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_n y_n^2$, which is obtained by an orthogonal transformation $X = \hat{P}Y$. Here $\hat{P}$ is known as normalized modal matrix and D is known as spectral matrix whose elements are the eigen values of matrix A.

Rank, Index, Signature and Nature of a Quadratic Form:

If the quadratic form (QF) $Q = X^T AX$ is reduced to the canonical form (CF) $Q' = Y^T DY$, then

1. ***Rank*** of a QF is the number terms in CF **or** the number of non-zero eigen values of the matrix A
2. ***Index*** of a QF is the number positive terms in CF **or** the number of positive eigen values of the matrix A.
3. ***Signature*** of a QF is the excess number of positive terms over the number of negative terms in CF **or** the excess number of positive eigen values over the number of negative eigen values of the matrix A.
4. ***Nature* of a QF**: A quadratic form $Q = X^T AX$ is said to be
   i) ***Positive definite*** if all the eigen values of A are positive.
   ii) ***Positive semi-definite*** if all the eigen values of A are non-negative (≥ 0) and at least one eigen value is 0
   iii) ***Negative definite*** if all the eigen values of A are negative.
   iv) ***Negative semi-definite*** if all the eigen values of A are non-positive (≤ 0) and at least one eigen value is 0
   v) ***Indefinite*** if some eigen values of A are positive and some are negative.

- The ***norm*** or length of a vector $X = [x_1, x_2, \ldots, x_n]^T$ is denoted by $\|X\|$ and is defined as $\|X\| = \sqrt{X^T X} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$
- Three vectors X_1, X_2 and X_3 are said to be pair wise orthogonal if $X_1^T X_2 = 0, X_2^T X_3 = 0$ an $X_3^T X_1 = 0.$
- The linearly independent eigen vectors corresponding to the distinct eigen values of a symmetric matrix A are always pair wise orthogonal.

Procedure to reduce Quadratic Form into Canonical Form by orthogonal transformation:

Let $Q = X^T AX$ be the QF in n variables $x_1, x_2, \ldots, x_n$.

Step1: Identify the symmetric matrix A associated with the $Q = X^T AX$, where $Y = [x_1, x_2, \ldots, x_n]^T$

Step 2: Find the eigen values of A, say, $\lambda_1, \lambda_2, \ldots, \lambda_n$

Step 3: Find the corresponding eigen vectors $X_1, X_2, \ldots, X_n$ such that they are pair wise orthogonal

Step 4: Find the normalized modal matrix $\hat{P} = \left[\frac{X_1}{\|X_1\|}, \frac{X_2}{\|X_2\|}, \ldots, \frac{X_n}{\|X_n\|} \right]$, which is always orthogonal.

Step 5: Let $X = \hat{P}Y \ldots (i)$ be the orthogonal transformation which transforms the given QF into CF, where $\hat{P}$ is known as the matrix of the transformation.

Step 6: By diagonalization, $D = \hat{P}^{-1} A\hat{P} = \hat{P}^T A\hat{P} \ldots (ii)$ $\left(\because \hat{P} \text{ is orthogonal}, \hat{P}^{-1} = \hat{P}^T\right)$.

$$\begin{aligned} \therefore\ X^T AX &= \left(\hat{P}Y\right)^T A\left(\hat{P}Y\right) = \left(Y^T \hat{P}^T\right) A\left(\hat{P}Y\right) \quad [\because \text{ By } (i)] \\ &= \left(Y^T \hat{P}^T\right) A\left(\hat{P}Y\right) \\ &= Y^T \left(\hat{P}^T A\hat{P}\right) Y \\ &= Y^T DY \quad [\because \text{ By } (ii)] \end{aligned}$$

Step 7: The required CF is $Y^T DY = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_n y_n^2$, where $Y = [y_1, y_2, \ldots, y_n]^T$

Multiple Choice Questions:

1. A real matrix $A = [a_{ij}]_{n\times n}$ is defined as $a_{ij} = \begin{cases} i & \text{, for } i = j \\ 0 & \text{, for } i \neq j \end{cases}$, then trace $(A) = \ldots\ldots$

A) $n(n+1)$ B) $n(n-1)$ C) $\frac{n(n-1)}{2}$ D) $\frac{n(n+1)}{2}$

Answer: **D**

2. If the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ is similar to matrix B, then sum of the eigen values of $B = \ldots\ldots$

A) 5 B) 6 C) 7 D) 9

Answer: **C**

3. Which of the following set represents the spectrum of a unitary matrix?

A) $\{\pm 1,\ 1 \pm i\}$ B) $\left\{\pm 1,\ \frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}\right\}$ C) $\{\pm 1,\ -1 \pm i\}$ D) $\{1 \pm i,\ -1 \pm i\}$

Answer: **B**

4. Let a and b be two real numbers such that $a^2 + b^2 = 1$. The eigen values of the non-singular matrix $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ are …….

A) $1, -1$ B) $2, -2$ C) $a, -a$ D) $b, -b$

Answer: **A**

5. If $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ is the eigen vector of matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ k & 2 & 3 \end{bmatrix}$ corresponding to the eigen value 3 then $k = \ldots\ldots$

A) -1 B) -2 C) 1 D) 2

Answer: **D**

6. If $1, -1, 2$ are eigen values of a matrix $A_{3\times 3}$ then trace$\left(A^2 - 3A + 5I\right) = \ldots\ldots$

A) 10 B) 12 C) 15 D) 18

Answer: **C**

7. If $A_{2\times 2}$ is a non-singular matrix such that trace $(A) = 5$ and trace$\left(A^2\right) = 9$ then $\det(A) = \ldots\ldots$

a) 7 b) 8 c) 5 d) 6

Answer: **B**

8. The matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ has three distinct eigen values and one of its eigen vectors is $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Which one of the following can be another eigen vector of A ?

B) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ B) $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ C) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ D) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Answer: **B**

9. If $M=\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ then $M^8-2M^7+2M^6-4M^5+3M^4-6M^3+2M^2=.....$

A) M B) $2M$ C) $3M$ D) $4M$

Answer: **D**

10. If $P=\begin{bmatrix}x & -3\\3 & 4\end{bmatrix}$ is a non-singular matrix with repeated eigen value and $x\in\mathbb{R}^+$ then $x=.....$

A) 2 B) 10 C) 4 D) 12

Answer: **B**

11. If $1,-2,3$ are eigen values of a matrix $A_{3\times3}$ then $A^{-1}=.....$

A) $\frac{1}{6}\left(5I-2A-A^2\right)$ B) $\frac{1}{6}\left(5I+2A+A^2\right)$ C) $\frac{1}{6}\left(5I+2A-A^2\right)$ D) $\frac{1}{6}\left(5I-3A-A^2\right)$

Answer: **C**

12. If p,q are index and signature of the QF $2x^2-3y^2-7z^2$ respectively then $p+q=.....$

A) 0 B) 2 C) 3 D) 4

Answer: **A**

UNIT-III: Sequences & Series

Sequence: A function $u:\mathbb{Z}^+\to\mathbb{R}$ is called a sequence of real numbers and is denoted by $\{u_n\}$ or $\langle u_n\rangle$.

Thus $\{u_n\}=u_1,u_2,u_3,\ldots,u_n,\ldots$

Here u_n is called the n^{th} term of the sequence $\{u_n\}$ and $u_1,u_2,u_3,\ .\ .\ .$ are called respectively first term, second term, third term etc.,

- The sequence $\{u_n\}$ denoted by $u_n=k\left(\in\mathbb{R}\right)$ is called a **constant sequence**.
- A sequence $\{u_n\}$ is said to be **bounded below** if there exists $k_1\in\mathbb{R}$ such that $k_1\le u_n\ \ \forall n\in\mathbb{Z}^+$, where k_1 is called the lower bound of the sequence $\{u_n\}$.
- If k_1 is a lower bound of the sequence $\{u_n\}$ then any number less than k_1 is a lower bound of $\{u_n\}$.
- If $\{u_n\}$ is bounded below, the greatest among the lower bounds of $\{u_n\}$ is called the greatest lower bound (**g.l.b**) of $\{u_n\}$.
- A sequence $\{u_n\}$ is said to be **bounded above** if there exists $k_2\in\mathbb{R}$ such that $u_n\le k_2\ \forall n\in\mathbb{Z}^+$, where k_2 is called the upper bound of the sequence $\{u_n\}$.
- If k_2 is an upper bound of the sequence $\{u_n\}$ then any number greater than k_2 is an upper bound of $\{u_n\}$.
- If $\{u_n\}$ is bounded above, the lowest among the upper bounds of $\{u_n\}$ is called the least upper bound (**l.u.b**) of $\{u_n\}$.
- A sequence $\{u_n\}$ is said to be **bounded** if there exists numbers k_1 & k_2 such that $k_1\le u_n\le k_2\ \forall n\in\mathbb{Z}^+$, otherwise $\{u_n\}$ is said to be unbounded.
- A sequence $\{u_n\}$ is said to be **monotonically increasing** if $u_{n+1}\ge u_n,\forall n$

i.e., $u_1 \le u_2 \le u_3 \le .....\le u_n \le u_{n+1} \le .....$

- A sequence $\{u_n\}$ is said to be **monotonically decreasing** if $u_{n+1} \le u_n, \forall n$

  i.e., $u_1 \ge u_2 \ge u_3 \ge .....\ge u_n \ge u_{n+1} \ge .....$

- A sequence $\{u_n\}$ is said to be **monotonic** if it is either monotonically increasing or monotonically decreasing.

Limit of a sequence: A real number l is said to be limit of $\{u_n\}$ if to each $\varepsilon > 0$, there exists $m \in \mathbb{Z}^+$ such that $|u_n - l| < \varepsilon, \forall n \ge m$.

If l is the limit of $\{u_n\}$, then we write $\lim_{n\to\infty} u_n = l$

Note:

(i) A sequence may have a unique limit or may have more than one limit or may not have a limit.

(iii) Limit of a sequence if it exists is unique

(iv) If the two sub-sequences $\{u_{2n}\}$ and $\{u_{2n-1}\}$ of sequence $\{u_n\}$ converges to the same limit l then $\{u_n\}$ also converges to l

(v) Every convergent sequence is bounded. But a bounded sequence need not be convergent.

For Example: The sequence $\{(-1)^n\} = -1, 1, -1, 1, -1, ...$ is bounded but not convergent.

Convergence, divergence and oscillation of a sequence:

- A sequence $\{u_n\}$ is said to be **convergent** if it has a finite limit *i.e.,* $\lim_{n\to\infty} u_n = l$ (finite value)

  For example: The sequence $\left\{\frac{1}{2^n}\right\}$ is convergent $\left(\because \lim_{n\to\infty} \frac{1}{2^n} = 0\right)$

- A sequence $\{u_n\}$ is said to be **divergent** if it has an infinite limit *i.e.,* $\lim_{n\to\infty} u_n = +\infty$ or $-\infty$

  For example: The sequence $\{n^2\}$ is divergent $\left(\because \lim_{n\to\infty} n^2 = +\infty\right)$

- If sequence $\{u_n\}$ neither converges to finite value nor diverges to $\pm\infty$ is said to be an **oscillatory**
- A bounded sequence which does not converge is said to **oscillate finitely**

  For example: The sequence $\{(-1)^n\}$ oscillates finitely since it is a bounded sequence and

  $$\lim_{n\to\infty} (-1)^n = \begin{cases} 1, & n \text{ is even} \\ -1, & n \text{ is odd} \end{cases}$$

- An unbounded sequence which does not diverge is said to **oscillate infinitely**

  For example: The sequence $\{(-1)^n n\}$ oscillates infinitely since it is an unbounded sequence and

  $$\lim_{n\to\infty} (-1)^n n = \begin{cases} +\infty, & n \text{ is even} \\ -\infty, & n \text{ is odd} \end{cases}$$

Infinite Series: If $\{u_n\}$ is a sequence of real numbers, then the expression $u_1 + u_2 + u_3 + ... + u_n + ...$ is called an infinite series *i.e.,* A series is a sum of the terms of the sequence.

The infinite series $u_1 + u_2 + u_3 + ... + u_n + ...$ is usually denoted by $\sum_{n=1}^{\infty} u_n$ or more briefly, by $\sum u_n$

Partial sums: If $\sum u_n$ is an infinite series, then $S_n = u_1 + u_2 + u_3 + ... + u_n$ is called the n^{th} partial sum of $\sum u_n$. Thus, the n^{th} partial sum of an infinite series is the sum of the first n terms.

- To every infinite series $\sum u_n$, there corresponds a sequence $\{S_n\}$ of its partial sums, where $S_1, S_2, S_3, ...$ are the first, second, third, …partial sums of the series

Behaviour of an infinite series: An infinite series $\sum u_n$ converges, diverges or oscillates (finitely or infinitely) according as the sequence $\{S_n\}$ of its partial sums converges, diverges or oscillates (finitely or infinitely)

- $\sum u_n$ is convergent if $\lim_{n\to\infty} S_n =$ finite
- $\sum u_n$ is divergent if $\lim_{n\to\infty} S_n = +\infty$ or $-\infty$
- $\sum u_n$ oscillates finitely if $\{S_n\}$ is bounded and not convergent
- $\sum u_n$ oscillates infinitely if $\{S_n\}$ is unbounded and not divergent

Necessary condition for convergence: If a series $\sum u_n$ is convergent, then $\lim_{n\to\infty} u_n = 0$

Preliminary test for divergence: If $\lim_{n\to\infty} u_n \neq 0$ then the series $\sum u_n$ is divergent

- A positive term series either converges or diverges to $+\infty$
- The geometric series $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + ...$

  (i) converges if $-1 < r < 1$ (ii) diverges if $r \geq 1$

  (iii) oscillates finitely if $r = -1$ (iv) oscillates infinitely if $r < -1$
- The $p-$harmonic series $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + ...$ converges if $p > 1$ and diverges if $p \leq 1$

Some useful standard limits:

(i) $\lim_{n\to\infty} \frac{1}{n^k} = 0$ for $k > 0$ (ii) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ (iii) $\lim_{n\to\infty} x^n = 0$ for $-1 < x < 1$

(iv) $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$ (v) $\lim_{n\to\infty} \left(1 + \frac{k}{n}\right)^n = e^k$ for any k

Series of positive terms: If all the terms of the series $\sum u_n$ are positive *i.e.,* $u_n > 0\ \forall n$, then $\sum u_n$ is called the series of positive terms.

Comparison test for series of positive terms: Comparison test for series of positive terms consists of "comparison" between a given (unknown) series $\sum u_n$ and a known auxiliary series $\sum v_n$ whose nature is known.

Comparison test for convergence: Let $\sum u_n$ and $\sum v_n$ be two series of positive terms such that $u_n \leq v_n\ \forall n$ and $\sum v_n$ converges then $\sum u_n$ also converges

Comparison test for divergence: Let $\sum u_n$ and $\sum v_n$ be two series of positive terms such that $u_n \geq v_n\ \forall n$ and $\sum v_n$ diverges then $\sum u_n$ also diverges

Limit form of the comparison test: Let $\sum u_n$ and $\sum v_n$ be two series of positive terms such that $\lim_{n\to\infty} \frac{u_n}{v_n} = l(\text{finite}) \neq 0$ then $\sum u_n$ and $\sum v_n$ both converge or diverge together

Note: Most often the geometric series $\sum_{n=0}^{\infty} r^n$ and the $p-$harmonic series $\sum \frac{1}{n^p}$ are chosen as a known auxiliary series $\sum v_n$ for comparison in case of above three comparison tests.

D'Alembert's Ratio Test: Let $\sum u_n$ be a series of positive terms such that $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l$(finite) then

$\sum u_n$ is said to be (i) convergent if $l > 1$

(ii) divergent if $l < 1$

and (iii) test fails when $l = 1$

Note: Apply Raabe's test when Ratio test fails

Raabe's Test: Let $\sum u_n$ be a series of positive terms such that $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$ (finite) then

$\sum u_n$ is said to be (i) convergent if $l > 1$

(ii) divergent if $l < 1$

and (iii) test fails when $l = 1$

Cauchy's nth Root Test: Let $\sum u_n$ be a series of positive terms such that $\lim_{n\to\infty} (u_n)^{1/n} = l$ (finite) then

$\sum u_n$ is said to be (i) convergent if $l < 1$

(ii) divergent if $l > 1$

and (iii) test fails when $l = 1$

Note: Apply Cauchy's n^{th} root test when u_n involves n^{th} powers of itself as whole

Cauchy's Integral Test: Let $\sum u_n = \sum f(n)$ be a series of positive terms such that $f(n)$ decreases as n increases and $\int_1^{\infty} f(x)dx = l$ then $\sum u_n$ is said to be (i) convergent if l is finite

(ii) divergent if l is infinite

Alternating series: A series in which the terms are alternate positive and negative is called an alternating series. Thus, the series $\sum (-1)^{n-1} v_n = v_1 - v_2 + v_3 - v_4 + ... + (-1)^{n-1} v_n + ...$, where $v_n > 0\ \forall n$, is an alternating series.

Leibnitz's Test: An alternating series of the form $\sum (-1)^{n-1} v_n$ is said to be convergent if $\{v_n\}$ is decreasing *i.e.*, $v_n \geq v_{n+1}\ \forall n$ and $\lim_{n\to\infty} v_n = 0$

- An alternating series $\sum u_n$ is said to be **absolutely convergent** if $\sum |u_n|$ is convergent
- An alternating series $\sum u_n$ is said to be **conditionally convergent** if $\sum u_n$ is convergent while $\sum |u_n|$ is divergent
- Every absolutely convergent series is convergent. But a convergent series need not be absolutely convergent.

Multiple Choice Questions:

1. Which of the following sequence is not bounded?

A) $\left\{\frac{1}{n}\right\}$ B) $\left\{1 + (-1)^n\right\}$ C) $\left\{(-1)^n\right\}$ D) $\left\{n + \frac{1}{n}\right\}$

Answer: **D**

2. Which of the following statement is FALSE?

A) Every convergent sequence is bounded B) Every bounded sequence is convergent

C) The sequence $\left\{(-1)^n\right\}$ oscillates finitely D) The sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ is convergent

Answer: **B**

3. Which of the following series is convergent?

A) $\sum \frac{1}{n}$ B) $\sum \left(\frac{3}{2}\right)^n$ C) $\sum \frac{1}{\sqrt{n}}$ D) $\sum \left(\frac{2}{3}\right)^n$

Answer: **D**

4. The n^{th} term of the series $\left(\frac{1}{4}\right)^1 + \left(\frac{2}{7}\right)^2 + \left(\frac{3}{10}\right)^3 + \dots\dots\dots$

A) $\left(\frac{n}{5n-1}\right)^n$ B) $\left(\frac{n}{3n+1}\right)^n$ C) $\left(\frac{n}{n+2}\right)^n$ D) $\left(\frac{n+1}{5n-1}\right)^n$

Answer: **B**

5. If $\sum v_n$ be the auxiliary series chosen to test the convergence of the series $\sum \frac{1}{n}\sin\left(\frac{1}{n}\right)$ then $v_n = \dots.$

A) $\frac{1}{\sqrt{n}}$ B) $\frac{1}{n}$ C) $\frac{1}{n^2}$ D) $\frac{1}{n\sqrt{n}}$

Answer: **C**

6. Which of the following test is best suited to test the convergence of the series $\sum \left(1+\frac{1}{n}\right)^{-n^2}$

A) Ratio test B) Raabe's test C) Comparison test D) Cauchy's n^{th} root test

Answer: **D**

7. The series $\sum \frac{1}{n^{\lambda-1}}$ converges if …….

A) $\lambda > 1$ B) $\lambda > 2$ C) $\lambda \le 1$ D) $\lambda \le 2$

Answer: **B**

8. The geometric series $\sum (-2)^n$ ……

A) converges B) diverges C) oscillates finitely D) oscillates infinitely

Answer: **D**

9. Which of the following test is best suited to test the convergence of the series $\sum \frac{(n!)^2}{(2n)!}$

A) Ratio test B) Leibnitz test C) Integral test D) Cauchy's n^{th} root test

Answer: **A**

10. The series $\sum \frac{4\cdot 7\cdot 10\cdot \cdots\cdot(3n+1)}{1\cdot 2\cdot 3\cdot \cdots\cdot n}$ is convergent if ......

A) $x < 3$ B) $x < \frac{1}{3}$ C) $x < \frac{1}{2}$ D) $x > \frac{1}{2}$

Answer: **B**

11. Which of the following series is absolutely convergent?

A) $\sum \frac{(-1)^{n-1}}{n}$ B) $\sum \frac{(-1)^{n-1}}{\sqrt{n}}$ C) $\sum \frac{(-1)^{n-1}}{n^2}$ D) $\sum \frac{(-1)^{n-1}}{4n-1}$

Answer: **C**

12. Which of the following series is conditionally convergent?

A) $\sum \frac{(-1)^{n-1}}{n}$ B) $\sum \frac{(-1)^{n-1}}{2^{n-1}}$ C) $\sum \frac{(-1)^{n-1}}{n^2}$ D) $\sum \frac{(-1)^{n-1}}{n^3-1}$

Answer: **B**

UNIT-IV: Beta & Gamma Functions and Mean Value Theorems

Beta & Gamma Functions: Many integrals which cannot be expressed in terms of elementary functions can be evaluated in terms of Beta and Gamma functions.

Gamma Function: If $n > 0$, then the definite integral $\int_0^\infty e^{-x} x^{n-1} dx$ is called the gamma function and it is denoted by $\Gamma(n)$ and read as gamma n.

$$\text{Thus } \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

Properties:

i) $\Gamma(1) = 1$

ii) $\Gamma(n) = (n-1)\Gamma(n-1)$ [Reduction Formula of $\Gamma(n)$]

iii) $\Gamma(n) = (n-1)!$, if n is a positive integer

iv) $\Gamma(n) = (n-1)(n-2)(n-3)....(n-k)\Gamma(n-k)$, where n is a positive fraction and $0 < (n-k) < 1$

v) $\Gamma(n) = \dfrac{\Gamma(n+k+1)}{n(n+1)(n+2)...(n+k)}$, where n is a negative fraction and $0 < (n+k+1) < 1$

vi) $\Gamma(n)$ is not defined for $n = 0, -1, -2, -3, ...$

vii) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Beta Function: If $m, n > 0$ then the definite integral $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ is called the beta function and is denoted by $\beta(m,n)$ *i.e.*, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

Properties:

i) Symmetry of Beta function: $\beta(m,n) = \beta(n,m)$

ii) Beta function in terms of trigonometric ratios: $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

Note: $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

Others forms of Beta Function:

Form -I: $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$

Form -II: $\beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$

Form -III: $\beta(m,n) = \frac{1}{(b-a)^{m+n-1}} \int_a^b (x-a)^{m-1}(b-x)^{n-1} dx$

Relationship between Beta and Gamma Functions: $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Result: $\Gamma(n)\Gamma(n-1) = \frac{\pi}{\sin n\pi}$ $(0 < n < 1)$

Example: (*i*) $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}$ (*ii*) $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}}$ (*iii*) $\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right) = 2\pi$

Mean Value Theorems:

- A function $f(x)$ is said to be continuous at a point $x = c$ if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$
- A function $f(x)$ is said to be continuous in the interval $[a,b]$ if it is continuous at every point of $[a,b]$
- A function $f(x)$ is said to be differentiable at a point $x = c$ if

  $$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$
- A function $f(x)$ is said to be differentiable in the interval $[a,b]$ if

(i) $f(x)$ is differentiable at every point of (a,b)

(ii) $\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}$ and $\lim_{x\to b^-}\frac{f(x)-f(b)}{x-b}$ exist

- If $f(x)$ is continuous in the interval $[a,b]$, then the graph of $y=f(x)$ is a continuous curve for the points in $[a,b]$
- If $f(x)$ is differentiable in the interval $[a,b]$, then there exist a unique tangent to the curve $y=f(x)$ at every point in $[a,b]$

Rolle's Theorem: Let a function $f:[a,b]\to\mathbb{R}$ be such that

(i) $f(x)$ is continuous in the interval $[a,b]$

(ii) $f(x)$ is differentiable in the interval (a,b) and

(iii) $f(a)=f(b)$ then there exist at least one value $c\in(a,b)$ such that $f'(c)=0$

Geometrical Interpretation: Under these assumptions of Rolle's theorem, there is at least one point on the curve $y=f(x)$ where the tangent is parallel to the $x-$ axis

Lagrange's Mean Value Theorem: Let a function $f:[a,b]\to\mathbb{R}$ be such that

(i) $f(x)$ is continuous in the interval $[a,b]$ and

(ii) $f(x)$ is differentiable in the interval (a,b) then there exist at least one value $c\in(a,b)$

such that $f'(c)=\frac{f(b)-f(a)}{b-a}$

Geometrical Interpretation: Under these assumptions of Lagrange's mean value theorem, there is at least one point on the curve $y=f(x)$ where the tangent is parallel to the chord joining the end points $A\big(a,f(a)\big)$ and $B\big(b,f(b)\big)$.

Cauchy's Mean Value Theorem: Let $f:[a,b]\to\mathbb{R}$, $g:[a,b]\to\mathbb{R}$ be two functions such that

(i) $f(x)$ and $g(x)$ are continuous in the interval $[a,b]$

(ii) $f(x)$ and $g(x)$ are differentiable in the interval (a,b) and

(iii) $g'(x)\neq 0\ \ \forall x\in(a,b)$ then there exist at least one value $c\in(a,b)$ such that

$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

Taylor's Theorem (Generalised Mean Value Theorem): Let a function $f:[a,b]\to\mathbb{R}$ be such that

(i) $f^{(n-1)}(x)$ is continuous on $[a,b]$

(ii) $f^{(n-1)}(x)$ is differentiable on (a,b) and $p\in\mathbb{Z}^+$ then there exist a point $c\in(a,b)$ such that

$$f(b)=f(a)+\frac{(b-a)}{1!}f'(a)+\frac{(b-a)^2}{2!}f''(a)+\frac{(b-a)^3}{3!}f'''(a)+.....+\frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+R_n$$

where $R_n=\frac{(b-a)^p(b-c)^{n-p}}{(n-1)!\,p}f^{(n)}(c)$ is called the remainder after n terms

Suppose $R_n\to 0$ as $n\to\infty$, then

$$f(x)=f(a)+\frac{(x-a)}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+.....+\frac{(x-a)^n}{n!}f^n(a)+.....\quad(1),$$

which is called **Taylor's series expansion** of $f(x)$ about $x=a$

Put $a=0$ in (1), we get

$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+....+\frac{x^n}{n!}f^n(0)+....$, which is called **Maclaurin's series expansion** of $f(x)$

Multiple Choice Questions:

1. $\Gamma\left(-\frac{1}{2}\right)=.........$

   A) $\sqrt{\pi}$ B) $2\sqrt{\pi}$ C) $-2\sqrt{\pi}$ D) $-\sqrt{\pi}$

   Answer: C

2. $\int_0^1 x^5(1-x)^3dx=........$

   A) $\beta(6,4)$ B) $\beta(5,3)$ C) $\beta(7,5)$ D) $\beta(6,3)$

   Answer: A

3. $\int_0^{\infty} x^6e^{-x}dx=......$

   A) $4!$ B) $6!$ C) $5!$ D) $7!$

   Answer: B

4. $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=........$

   A) $2\sqrt{\pi}$ B) $-2\sqrt{\pi}$ C) $\pi\sqrt{2}$ D) $-\pi\sqrt{2}$

   Answer: C

5. $\int_0^{\infty}\frac{x^{10}-x^{18}}{(1+x)^{30}}dx=.......$

   A) 0 B) 1 C) 2 D) 3

   Answer: A

6. $\int_0^1\left(\log_e\frac{1}{x}\right)^3dx=......$

   A) 24 B) 6 C) 12 D) 10

   Answer: B

7. If $f(x)=x^4+x^2-2$ satisfies the conditions of Rolle's theorem on $[a,b]$ and $a=-1$ then $b=....$

   A) 2 B) -2 C) 1 D) -1

   Answer: C

8. The Lagrange's mean value theorem is satisfied for $f(x)=x^3+5x$ in $[1,4]$ at a value of $x=....$

   A) $\sqrt{5}$ B) $\sqrt{6}$ C) $\sqrt{7}$ D) $\sqrt{11}$

   Answer: C

9. If $a+b+c=0$ then one of the roots of the equation $3ax^2+2bx+c=0$ lies in the interval ….

   A) $(-1,1)$ B) $(0,1)$ C) $(1,2)$ D) $(-1,0)$

   Answer: B

10. The value of 'c' of Cauchy's mean value theorem for $f(x)=e^x$ and $g(x)=e^{-x}$ in $[2,6]$ is

    A) 4 B) 5 C) 3.5 D) 3

    Answer: A

11. Which of the following theorem is known as higher mean value theorem?

    A) Rolle's theorem B) Lagrange's mean value theorem

C) Cauchy's mean value theorem D) Taylor's theorem

Answer: D

12. Maclaurin's series expansion of $\tan^{-1}x$ is …..

A) $x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+......$ B) $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+......$

A) $1+\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+......$ B) $1-\frac{x^2}{2}+\frac{x^4}{4}-\frac{x^6}{6}+......$

Answer: B

UNIT-V: Functions of Several Variables

Partial derivative: A partial derivative of a function of several variables is the ordinary derivative with respect to one of the variables when all the remaining variables are held constant.

Let $z=f(x,y)$ be a function of two variables x,y.

- The derivative of z with respect to x, treating y as constant, is called the partial derivative of z with respect to x and is denoted by $\frac{\partial z}{\partial x}$ or z_x. Thus $\frac{\partial z}{\partial x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$

Similarly, the derivative of z with respect to y, treating x as constant, is called the partial derivative of z with respect to y and is denoted by $\frac{\partial z}{\partial y}$ or z_y. Thus $\frac{\partial z}{\partial y}=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$

Partial derivatives of second order, of a function $f(x,y)$ are calculated by successive differentiation. Thus if $z=f(x,y)$ then

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=z_{xx},\ \frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=z_{xy},\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=z_{xy}\ \text{and}\ \frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=z_{yy}$$

Note: A function of 2 variables has 2 first order partial derivatives, 2^2 second order partial derivatives, 2^3 third order partial derivatives and so on.

Total derivative: Total differential of a function u of three variables x,y,z is denoted by du and is defined as $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$

Chain rule: If $u=f(x,y,z)$, where x,y,z are functions of a variable t then the total derivative of u is defined as $\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt}$ . . . (i)

Corollary: If $t=x$, (i) becomes, $\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}+\frac{\partial u}{\partial z}\frac{dz}{dx}$

Differentiation of implicit function: If $f(x,y)=c$ be an implicit relation between x and y which defines as a differentiable function of x, then $\frac{dy}{dx}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}$

Jacobians: Jacobians are functional determinants (whose elements are functions) which are very useful in transformation of variables from cartesian to polar, cylindrical and spherical coordinates in multiple integrals.

Definition: If u and v are functions of two independent variables x and y, then the determinant $\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$ is called Jacobian of u,v with respect to x,y and is written as $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$

Similarly, the Jacobian of u,v,w with respect to x,y,z is $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

Properties of Jacobians:

i) If $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$ then $JJ' = 1$

ii) Chain rule of Jacobians: If u,v are functions of r,s and r,s are functions of x,y then

$$\frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(x,y)}$$

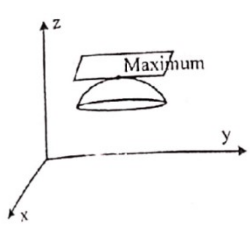
Functional Dependence: Let $u = f(x,y), v = g(x,y)$ be two given differentiable functions of the two independent variables x and y. Suppose these functions u and v are connected by a relation $F(u,v) = 0$, where F is differentiable. We say that u and v are functionally dependent on one another if u_x, u_y, v_x and v_y not all zero simultaneously.

- If u,v,w be functions of three independent variables x,y,z then u,v,w are functionally dependent (related) if and only if $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

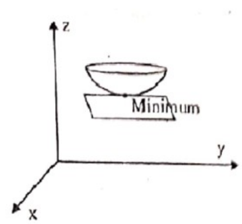
Maxima and Minima of functions of two variables:

- A function $f(x,y)$ is said to have a maximum value at $x = a, y = b$ if $f(a,b) > f(a+h,b+k)$ for all positive or negative small values of h and k.
- A function $f(x,y)$ is said to have a minimum value at $x = a, y = b$ if $f(a,b) < f(a+h,b+k)$ for all positive or negative small values of h and k.

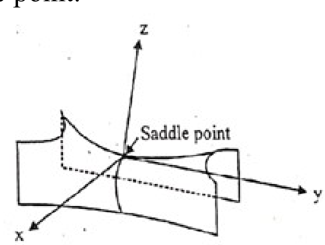
Geometrically $z = f(x,y)$ represents a surface. The maximum is a point on the surface (hill top) from which the surface descends (comes down) in every direction towards the $xy-$plane (Fig (a)). The minimum is the bottom of depression from which the surface ascends (climbs up) in every direction (Fig (b)).Besides these, we have such a point of the surface, where the tangent plane is horizontal and the surface looks like leather seat on horse's back (Fig (c)) which falls displacement in certain directions and rises for displacements in another directions. Such a point is called a saddle point.



(a)



(b)



(c)

Conditions for Maxima and Minima of functions of *f*(*x*, *y*):

The necessary conditions for $f(x,y)$ to have a maximum or minimum at (a,b) are that

$$\frac{\partial f(a,b)}{\partial x} = 0, \frac{\partial f(a,b)}{\partial y} = 0$$

Stationary point: The point (a,b) is called a stationary point if $f_x(a,b) = 0, f_y(a,b) = 0$

Stationary value: $f(a,b)$ is said to be a stationary value of $f(x,y)$ if $f_x(a,b) = 0, f_y(a,b) = 0$

i.e., the function is stationary at (a,b).

Extreme value: A maximum or minimum value of a function is called its extreme value.

Working rule to find the maximum and minimum values of $f(x,y)$:

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and equate each to zero. Solve these as simultaneous equations in x and y.

   Let $(a_1,b_1),(a_2,b_2),(a_3,b_3),....$ be the pairs of values and are called stationary points of $f(x,y)$

2. Calculate $r=\frac{\partial^2 f}{\partial x^2}, s=\frac{\partial^2 f}{\partial x \partial y}, t=\frac{\partial^2 f}{\partial y^2}$ at each of the stationary point.

3. (*i*) If $rt-s^2>0$ and $r<0$ at (a_1,b_1) then f has a maximum at (a_1,b_1) and $f_{\max}=f(a_1,b_1)$

   (*ii*) If $rt-s^2>0$ and $r>0$ at (a_1,b_1) then f has a minimum at (a_1,b_1) and $f_{\min}=f(a_1,b_1)$

   (*iii*) If $rt-s^2<0$ at (a_1,b_1) then f has neither maximum nor minimum at (a_1,b_1) i.e., (a_1,b_1) is a saddle point.

   (*iv*) If $rt-s^2=0$ at (a_1,b_1), no conclusion can be drawn about maximum or minimum and it needs further investigation

Similarly examine the pair of values $(a_2,b_2),(a_3,b_3),....$ one by one

Lagrange's Method of undetermined multipliers: Let $f(x,y,z)$ be a function of three variables x,y,z which are connected by the relation $\phi(x,y,z)=0$ . . . . (1)

Consider the Lagrangian function $F(x,y,z)=f(x,y,z)+\lambda\phi f(x,y,z),$ where λ is the Lagrangian multiplier

For maxima or minima of $F(x,y,z)$, we have

$$\frac{\partial F}{\partial x}=0 \text{ i.e., } \frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0 \quad .... (2)$$

$$\frac{\partial F}{\partial y}=0 \text{ i.e., } \frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0 \quad .... (3)$$

$$\frac{\partial F}{\partial z}=0 \text{ i.e., } \frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0 \quad .... (4)$$

On solving (1),(2),(3) and (4),we can find the values of x,y,z and λ for which $f(x,y,z)$ has stationary value.

Note: This method gives us the stationary value of a given function. But we cannot determine the nature of stationary points. However, this can be decided by physical or geometrical considerations.

Multiple Choice Questions:

1. If $z=x^2+y^2$, where $x=e^t\sin t,\ y=e^t\cos t$ then $\frac{dz}{dt}$ at $t=0$ is .....

   A) 2 B) 4 C) 6 D) 8

   Answer: A

2. If $u=e^{x^2+y^2}$ then $\frac{\partial^2 u}{\partial x \partial y}=......$

   A) xyu B) $2xyu$ C) $4xyu$ D) $8xyu$

   Answer: C

3. If $ye^{xy}=\cos x$ then $\frac{dy}{dx}$ at $(0,1)=.....$

   A) 1 B) -1 C) 0 D) 2

   Answer: B

4. Let $w=\phi(x,y)$, where x,y are functions of t. Then, according to chain rule, $\frac{dw}{dt}=....$

A) $\frac{\partial x}{\partial t}\frac{d\phi}{dx}+\frac{\partial y}{\partial t}\frac{d\phi}{dy}$ B) $\frac{\partial \phi}{\partial x}\frac{dx}{dt}+\frac{\partial \phi}{\partial y}\frac{dy}{dt}$

C) $\frac{d\phi}{dx}\frac{dx}{dt}+\frac{d\phi}{dy}\frac{dy}{dt}$ D) $\frac{\partial \phi}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial \phi}{\partial y}\frac{\partial y}{\partial t}$

Answer: B

5. If $u = x(1+y),\ v = y(1+x)$ then the value of $\frac{\partial(u,v)}{\partial(x,y)}$ at $x=1$ & $y=1$ is .....

A) 1 B) 2 C) 3 D) 4

Answer: C

6. If $u=\frac{x}{y},\ v=\frac{x+y}{x-y}$ are functionally related then the functional relation between them is ….

A) $v=\frac{u+1}{u-1}$ B) $u=\frac{v+1}{v-1}$ C) $u=\frac{1+v}{1-v}$ D) $v=\frac{1+u}{1-u}$

Answer: A

7. If $u=2x-y,\ v=y+2z,\ w=x-3z$ then the value of $\frac{\partial(u,v,w)}{\partial(x,y,z)}=$......

A) -2 B) 4 C) 3 D) 2

Answer: B

8. Which of the following statement is not **TRUE**?

A) Two functions $u=f(x,y), v=g(x,y)$ are functionally dependent if $\frac{\partial(u,v)}{\partial(x,y)}=0$

B) If $\frac{\partial(u,v)}{\partial(x,y)}\neq 0$ then $u=f(x,y), v=g(x,y)$ are functionally independent

C) The functions $u=e^x \sin y, v=e^x \cos y$ are functionally dependent

D) If $J=\frac{\partial(u,v,w)}{\partial(x,y,z)}$ and $J^*=\frac{\partial(x,y,z)}{\partial(u,v,w)}$ then $J\times J^*=1$

Answer: C

9. Which of the following is a stationary point of $f(x,y)=x^2+y^2-6x+12$

A) $(0,3)$ B) $(3,0)$ C) $(0,2)$ D) $(0,0)$

Answer: B

10. If $(0,0),(2,0)$ are extreme points of $f(x,y)=x^3+3xy^2-3x^2-3y^2+7$ then $f_{\min.}=$......

A) 7 B) -8 C) 2 D) 3

Answer: D

11. If $r=f_{xx}(a,b),\ s=f_{xy}(a,b),\ t=f_{yy}(a,b)$, then $f(x,y)$ will have maximum at (a,b) if

A) $(rt-s^2)>0$ and $r>0$ B) $(rt-s^2)>0$ and $r<0$

C) $(rt-s^2)<0$ and $r>0$ D) $(rt-s^2)<0$ and $r<0$

Answer: B

12. If $r=f_{xx}(a,b),\ s=f_{xy}(a,b),\ t=f_{yy}(a,b)$ & $(rt-s^2)<0$ then $f(x,y)$ will have

A) maximum at (a,b) B) minimum at (a,b)

C) neither maximum nor minimum at (a,b) D) either maximum or minimum at (a,b)

Answer: C

SOME USEFUL FORMULAE

1. TRIGONOMETRY

- $\sin^2 x + \cos^2 x = 1$
- $\sec^2 x - \tan^2 x = 1$
- $\text{cosec}^2 x - \cot^2 x = 1$
- $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
- $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
- $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$
- $\sin 2x = 2 \sin x \cos x = \dfrac{2 \tan x}{1 + \tan^2 x}$
- $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$
- $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$
- $\sin x \cos y = \dfrac{1}{2}[\sin(x + y) + \sin(x - y)]$
- $\cos x \sin y = \dfrac{1}{2}[\sin(x + y) - \sin(x - y)]$
- $\cos x \cos y = \dfrac{1}{2}[\cos(x + y) + \cos(x - y)]$
- $\sin x \sin y = \dfrac{1}{2}[\cos(x - y) - \cos(x + y)]$
- $\sin 3x = 3 \sin x - 4 \sin^3 x$
- $\cos 3x = 4\cos^3 x - 3\cos x$
- $\sinh x = \dfrac{e^x - e^{-x}}{2}$
- $\cosh x = \dfrac{e^x + e^{-x}}{2}$
- $\sinh 2x = 2 \sinh x \cosh x$
- $\cosh 2x = \cosh^2 x + \sinh^2 x$
- $\cosh^2 x - \sinh^2 x = 1$
- $\sinh^2 x = \dfrac{1}{2}(\cosh 2x - 1)$
- $\cosh^2 x = \dfrac{1}{2}(\cosh 2x + 1)$
- $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$
- $\cosh 3x = 4\cosh^3 x - 3\cosh x$
- $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
- $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
- $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
- $e^{iax} = \cos ax + i \sin ax$
- $\text{Re}(e^{iax}) = \cos ax$
- $\text{Im}(e^{iax}) = \sin ax$
- $\sin ix = i \sinh x$
- $\cos ix = \cosh x$
- $\tan ix = i \tanh x$
- $\sinh 0 = \dfrac{e^0 - e^{-0}}{2} = 0$
- $\cosh 0 = \dfrac{e^0 + e^{-0}}{2} = 1$
- $\sinh^{-1}(x/a) = \log\left(x + \sqrt{x^2 + a^2}\right)$
- $\cosh^{-1}(x/a) = \log\left(x + \sqrt{x^2 - a^2}\right)$
- $\tanh^{-1}(x/a) = \dfrac{1}{2}\log\left(\dfrac{a + x}{a - x}\right)$

- If n is a positive integer then,

  $\sin n\pi = \sin 2n\pi = \sin(2n \pm 1) = 0$, $\cos n\pi = (-1)^n$, $\cos 2n\pi = 1$, $\cos(2n \pm 1) = -1$

- $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$
- $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$
- $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$
- $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

2. DIFFERENTIATION

- $\frac{d}{dx}(uv) = uv' + u'v$
- $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$
- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(a^x) = a^x \log a$
- $\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$
- $\frac{d}{dx}(\log_e x) = \frac{1}{x}\log_e e = \frac{1}{x}$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
- $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
- $\frac{d}{dx}(\sinh x) = \cosh x$
- $\frac{d}{dx}(\cosh x) = \sinh x$
- $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
- $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
- $\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$
- $\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$
- $\frac{d}{dx}\left[\sin^{-1}\left(\frac{x}{a}\right)\right] = \frac{1}{\sqrt{a^2 - x^2}}$
- $\frac{d}{dx}\left[\cos^{-1}\left(\frac{x}{a}\right)\right] = -\frac{1}{\sqrt{a^2 - x^2}}$
- $\frac{d}{dx}\left[\tan^{-1}\left(\frac{x}{a}\right)\right] = \frac{a}{a^2 + x^2}$
- $\frac{d}{dx}\left[\sinh^{-1}\left(\frac{x}{a}\right)\right] = \frac{1}{\sqrt{a^2 + x^2}}$
- $\frac{d}{dx}\left[\cosh^{-1}\left(\frac{x}{a}\right)\right] = \frac{1}{\sqrt{x^2 - a^2}}$
- $\frac{d}{dx}\left[\tanh^{-1}\left(\frac{x}{a}\right)\right] = \frac{a}{a^2 - x^2}$
- $\frac{d}{dx}[\log f(x)] = \frac{f'(x)}{f(x)}$
- $\frac{d}{dx}\left[e^{f(x)}\right] = e^{f(x)} f'(x)$
- $\frac{d}{dx}\left[\sqrt{f(x)}\right] = \frac{f'(x)}{2\sqrt{f(x)}}$
- $\frac{d}{dx}\left[\frac{1}{f(x)}\right] = -\frac{f'(x)}{[f(x)]^2}$
- $\frac{d}{dx}\left\{[f(x)]^{n+1}\right\} = (n+1)[f(x)]^n f'(x)$

3. INTEGRATION

- $\int x^n dx = \frac{x^{n+1}}{n+1}, n \neq -1$
- $\int e^x dx = e^x$
- $\int a^x dx = \frac{a^x}{\log a}, a \neq 1$

- $\int \frac{1}{x} dx = \log x$
- $\int \sin x dx = -\cos x$
- $\int \cos x dx = \sin x$
- $\int \sec^2 x dx = \tan x$
- $\int \sec x \tan x \, dx = \sec x$
- $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x$
- $\int \operatorname{cosec}^2 x \, dx = -\cot x$
- $\int \sinh x \, dx = \cosh x$
- $\int \cosh x \, dx = \sinh x$
- $\int \operatorname{sech}^2 x \, dx = \tanh x$
- $\int \operatorname{cosech} x \coth x dx = -\operatorname{cosech} x$
- $\int \operatorname{cosech}^2 x dx = \coth x$
- $\int \operatorname{sech} x \tanh x dx = -\operatorname{sech} x$
- $\int \sec x \, dx = \log|\sec x + \tan x| = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right|$
- $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| = \log\left|\tan\frac{x}{2}\right|$
- $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) = -\cos^{-1}\left(\frac{x}{a}\right)$
- $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
- $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) = \log\left(x + \sqrt{a^2 + x^2}\right)$
- $\int [f(x)]^n f'(x) \, dx = \frac{[f(x)]^{n+1}}{n+1}, n \neq -1$
- $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) = \log\left(x + \sqrt{x^2 - a^2}\right)$
- $\int \frac{f'(x)}{f(x)} = \log|f(x)|$
- $\int e^{f(x)} f'(x) dx = e^{f(x)}$
- $\int \frac{f'(x)}{\sqrt{f(x)}} = 2\sqrt{f(x)}$
- $\int \frac{dx}{\sqrt{x}} = 2\sqrt{x}$
- $\int \cot x dx = \log|\sin x|$
- $\int \tan x \, dx = \log|\sec x| = -\log|\cos x|$
- $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$
- $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$
- $\int e^x [f(x) + f'(x)] dx = e^x f(x)$

Integration by parts: If u and v are functions of x, then $\int uv \, dx = u\int v \, dx - \int\left\{\frac{du}{dx}\int v \, dx\right\}dx$ where u = first function, v = second function. Here the first function is the function which comes first in the word ILATE.

- $\int e^{ax} f(x) dx = \frac{e^{ax}}{a}\left[f(x) - \frac{f'(x)}{a} + \frac{f''(x)}{a^2} - \frac{f'''(x)}{a^3} + \ldots\right]$
- $\int e^x x \, dx = e^x (x - 1)$
- $\int e^{-x} x \, dx = -e^{-x}(x + 1)$
- $\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$
- $\int \sqrt{a^2 + x^2} \, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)$
- $\int \sqrt{x^2 - a^2} \, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)$

- $\int e^{ax}\sin(bx+c)dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx+c) - b\cos(bx+c)]$

- $\int e^{ax}\cos(bx+c)dx = \frac{e^{ax}}{a^2+b^2}[a\cos(bx+c) + b\sin(bx+c)]$

Properties of definite integrals:

- $\int_a^b f(x)dx = \int_a^b f(y)dy$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^a f(x)dx = 0$

- If $a < c < b$ then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

- $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \ldots + \int_{c_n}^b f(x)dx$, where $a < c_1 < c_2 < \ldots < c_n < b$

- $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$
- $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

- $\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx & \text{if } f(-x) = f(x) \\ 0 & \text{if } f(-x) = -f(x) \end{cases}$
- $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$

- $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \left(\frac{n-1}{n}\right)\cdot\left(\frac{n-3}{n-2}\right)\cdot\left(\frac{n-5}{n-4}\right)\ldots\ldots\frac{1}{2}\cdot\frac{\pi}{2}, & \text{if } n \text{ is even} \\ \left(\frac{n-1}{n}\right)\cdot\left(\frac{n-3}{n}\right)\cdot\left(\frac{n-5}{n-4}\right)\ldots\ldots\frac{2}{3}\cdot 1, & \text{if } n \text{ is odd} \end{cases}$

- $\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{[(m-1)(m-3)(m-5)\ldots\ldots][(n-1)(n-3)(n-5)\ldots\ldots]}{[(m+n)(m+n-2)(m+n-4)\ldots\ldots]} \times k$

  where k is $\pi/2$ if m and n are even otherwise k is 1

- $\int_0^\infty \frac{\sin mx}{x}dx = \frac{\pi}{2}$ $(m > 0)$
- $\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}$
- $\int_{-\infty}^0 e^{-x^2}dx = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

- $\int_0^\infty e^{-ax}\sin(bx+c)dx = \frac{b}{a^2+b^2}$
- $\int_0^\infty e^{-ax}\cos(bx+c)dx = \frac{a}{a^2+b^2}$

- **Leibnitz's rule of integration by parts:**

  $$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\ldots$$

  where superscript (′) denotes differentiation, i.e., u''' denotes differentiation of u thrice and subscript number denotes number of times of integration i.e., v_2 denotes integration of v twice

4. SERIES

- $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$
- $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots$
- $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots\ldots$
- $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\ldots$
- $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$
- $\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\ldots\right)$
- $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$
- $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{6}$
- If n is a rational number then

  (i) $(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}x^3 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^4 + \ldots\ldots$

  (ii) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{1\cdot 2}x^2 + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}x^3 + \frac{n(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3\cdot 4}x^4 + \ldots\ldots$

  (iii) $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{1\cdot 2}x^2 - \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}x^3 + \frac{n(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3\cdot 4}x^4 - \ldots\ldots$
- $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots\ldots$
- $(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots\ldots$
- $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \ldots\ldots$
- $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots\ldots$
- $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12}$
- $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$

5. ALGEBRA

(i) Logarithms

- Natural logarithm $\log_e x$ has base e and is inverse of e^x
- Common logarithm $\log_{10} x$ has base 10 and is inverse of 10^x

- $\log_a 1 = 0$ • $\log_a a = 1$ • $\log_a mn = \log_a m + \log_a n$ • $\log_a (m/n) = \log_a m - \log_a n$
- $\log m^n = n \log m$ • $e^{\log f(x)} = f(x)$

(ii) Progressions

- Numbers $a, a+d, a+2d, ...$ are said to be in Arithmetic Progression (A.P). Its n^{th} term $T_n = a + (n-1)d$ and the sum of first n terms $S_n = \frac{n}{2}[2a + (n-1)d]$
- Numbers $a, ar, ar^2, ...$ are said to be in Geometric Progression (G.P). Its n^{th} term $T_n = ar^{n-1}$ and the sum of first n terms $S_n = \frac{a(1-r^n)}{1-r}$ • $S_\infty = \frac{a}{1-r} (r < 1)$
- If a and b be two numbers then their Arithmetic Mean (A.M) $= \frac{a+b}{2}$ Geometric Mean (G.M) $= \sqrt{ab}$ and Harmonic Mean (H.M) $= \frac{2ab}{a+b}$
- For the first n natural numbers 1, 2, 3,..., n

$$\sum_1^n k = \frac{n(n+1)}{2}, \sum_1^n k^2 = \frac{n(n+1)(2n+1)}{6}, \sum_1^n k^3 = \frac{n^2(n+1)^2}{4} = \left(\sum_1^n k\right)^2$$

(iii) Permutations and Combinations

- ${}^nP_r = \frac{n!}{(n-r)!}$ • ${}^nC_r = \frac{n!}{(n-r)!\,r!}$ • ${}^nC_r = {}^nC_{n-r}$
- ${}^nC_0 = 1 = {}^nC_n$ • $n! = n(n-1)(n-2)......3 \cdot 2 \cdot 1$ • $0! = 1$
- ${}^nC_r = {}^nC_s$ then either $r = s$ or $r + s = n$

(iv) Binomial Theorem

- If n is an integer then $(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + ... + {}^nC_r x^{n-r} y^r + ... + {}^nC_n y^n$
- The general term in $(x+y)^n$ is $T_{r+1} = {}^nC_r x^{n-r} y^r$
- ${}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$ • ${}^nC_0 + {}^nC_2 + {}^nC_4 + ... = {}^nC_1 + {}^nC_3 + {}^nC_5 + ... = 2^{n-1}$